Lorentzian regularization and the problem of point-like particles in general relativity

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Abstract

The two purposes of the paper are (1) to present a regularization of the self-field of point-like particles, based on Hadamard's concept of "partie finie", that permits in principle to maintain the Lorentz covariance of a relativistic field theory, (2) to use this regularization for defining a model of stress-energy tensor that describes point-particles in post-Newtonian expansions (e.g. 3PN) of general relativity. We consider specifically the case of a system of two point-particles. We first perform a Lorentz transformation of the system's variables which carries one of the particles to its rest frame, next implement the Hadamard regularization within that frame, and finally come back to the original variables with the help of the inverse Lorentz transformation. The Lorentzian regularization is defined in this way up to any order in the relativistic parameter $1/c^2$. Following a previous work of ours, we then construct the delta-pseudo-functions associated with this regularization. Using an action principle, we derive the stress-energy tensor, made of delta-pseudo-functions, of point-like particles. The equations of motion take the same form as the geodesic equations of test particles on a fixed background, but the role of the background is now played by the regularized metric.

I. INTRODUCTION

In recent years, the problem of the dynamics of gravitationally interacting compact objects in general relativity has received a lot of attention. This is due in part to the interest of the theoretical problem in its own, and in part to the ongoing development of laser-interferometric detectors for observing gravitational radiation. In the absence of an exact solution of the problem, one has recourse to successive post-Newtonian approximations (formal expansions in powers of $1/c$). Within such approximations, it makes sense to model the compact objects with some "point-like particles", exactly as we do in a standard way within the Newtonian theory. However, the self-field of point-particles is infinite at the very location of a particle, and thus must be somehow regularized. The regularization is quite straightforward in the Newtonian theory, but it becomes non-trivial when going to high post-Newtonian approximations. Dealing with this problem, the present authors [1] developed a method for regularizing the infinite self-field of point-particles, which is based on the concept of "partie finie", in the sense of Hadamard [2,3], of a singular function at the place of one of its singular points (see e.g. [4–7] for entries to the mathematical literature). We know that the Hadamard regularization yields the correct result for the equations of motion of two particles up to the so-called second and half post-Newtonian (2.5PN) approximation, corresponding to the order $1/c^5$ beyond the Newtonian acceleration. Indeed, the problem has been completely solved at that order [8–19]; notably some derivations make use of this regularization (e.g. [12,19]). In the present state of the art, we are concerned with the 3PN (or $1/c^6$) approximation [20–25]. In fact, starting at this high post-Newtonian order, the regularization may become physically incomplete because of the appearance of an undetermined coefficient in the equations of motion [20–25].

The Hadamard regularization, investigated in [1], is performed in a three-dimensional Euclidean space with generic point $\mathbf{x} \in \mathbb{R}^3$, which is viewed as the spatial hypersurface labelled by $t =$const in a global coordinate system $\{t, \mathbf{x}\}$ covering the whole space-time. In particular, the regularization involves a spatial average, performed at $t =$const, over the direction of approach to the singularity. As such a regularization makes use of a preferred spatial hypersurface $t =$const, it is clearly incompatible with the framework of special or general relativity, which embodies a global Lorentz (or Poincaré) frame invariance. Notably, we expect that the post-Newtonian equations of motion of point-like particles in harmonic coordinates (which we recall preserve the global Lorentz invariance) should exhibit at some stage a violation of the Lorentz invariance due to the latter regularization. The fact is that the breakdown of the Lorentz invariance due to the regularization occurs only at the very high 3PN approximation. Untill the 2.5PN order, it is sufficient to regularize within a preferred slice $t =$const of the harmonic coordinate system to obtain some Lorentz-invariant equations of motion [19].

The first purpose of this paper is to define a regularization *à la* Hadamard [2,3] that is compatible with the Lorentz structure of a relativistic field theory. This completes the definition, proposed in [1], of a specific version of the Hadamard regularization (based notably

on a particular class of pseudo-functions). To achieve this purpose, we shall simply perform the standard Hadamard regularization within the hypersurface that is geometrically orthogonal, in the sense of the Minkowski metric, to the four-velocity of the particle. In separate papers [24,25], we apply the latter "Lorentzian" regularization (together with the distributional derivatives introduced in [1]), to the computation of the binary equations of motion at the 3PN order in harmonic coordinates, and find that, indeed, it permits the preserving of their Lorentz invariance (in some case at the price of adjusting some parameter). A different approach to the problem of incorporating the Lorentz invariance in the 3PN equations of motion consists of deriving a generic regularized dynamics, within the ADM-Hamiltonian formalism of general relativity, involving an arbitrary regularization parameter, and to determine this parameter uniquely by requiring the Lorentz invariance [23]. (See Section 2 in [25] for a discussion on our point-mass regularization and its relation to [23].)

All-over the paper, we assume the existence of a preferred Minkowski metric, as selected for instance by the condition of harmonic coordinates in general relativity, with respect to which the trajectories of particles are represented by accelerated world lines like in special relativity. Most of our investigation is valid not only in the case of the gravitational field but also for any Lorentz-tensor field propagating on the Minkowski background. Furthermore, we shall define the Lorentzian regularization in a sense of formal expansion series in $1/c^2$; so that, all the formulas in the paper will be given by some infinite series of relativistic corrections when c tends toward infinity. This is all what we need for the derivation of the equations of motion to the 3PN order [24,25].

Since we are interested in the application to the motion of two particles, we shall define the regularization around one of the particles (say particle 1), and shall consider that its acceleration is purely due to particle 2. [However our definitions could be generalized to a system of N particles.] Notice that the particle 2 enters this regularization scheme through the Lorentz transformation of its own variables to the rest frame of particle 1, and the replacement of the acceleration of 1 in terms of the equations of the binary motion. In general, working at some given relativistic order, we shall need to know the equations of motion up to a lower order only, therefore giving us the possiblity of an iterative process. In this paper, we always assume that we know the relevant equations of motion at this order, and that these are Lorentz-invariant.

Our second purpose is to derive an expression, compatible with the latter regularization, for the stress-energy tensor of point-like particles in post-Newtonian expansions of general relativity. Thanks to this regularization, we are able to give a sense to the value of the metric coefficients at the very location of the particle. Our basic assumption is that the matter action is the same as for *test* particles moving on a prescribed background gravitational field, except that the metric at the location of the particles is replaced by its regularized value in the sense of the (Lorentzian) regularization. From this assumption, we prove that the Dirac measure in the stress-energy tensor must be replaced by a certain generalized function defined by means of the Hadamard prescription. In the case of two particles (the generalization to N particles is immediate), we obtain

$$T^{\mu\nu}_{\rm particle} = \frac{m_1 c\, v_1^\mu v_1^\nu}{\sqrt{-[g_{\rho\sigma}]_1 v_1^\rho v_1^\sigma}}\, \mathrm{Pf}\left(\frac{\Delta(\mathbf{x}-\mathbf{y}_1)}{\sqrt{g(t,\mathbf{x})}}\right) + 1 \leftrightarrow 2\;, \tag{1.1}$$

where m_1 is the mass of the particle 1, and $v_1^\mu = (c, \mathbf{v}_1)$ its coordinate velocity, i.e. $\mathbf{v}_1 = d\mathbf{y}_1/dt$, $\mathbf{y}_1 = \mathbf{y}_1(t)$ being the trajectory parametrized by the coordinate time t (the symbol $1 \leftrightarrow 2$ denotes the same expression but corresponding to the second particle). The notation $[g_{\rho\sigma}]_1$ means that the metric $g_{\rho\sigma}(t,\mathbf{x})$ is to be computed at the point $\mathbf{x} = \mathbf{y}_1(t)$ following the regularization (of course $[g_{\rho\sigma}]_1$ depends on the positions and velocities of both particles 1 and 2). Note that the first factor in (1.1) is a mere function of time t. The second factor $\mathrm{Pf}\left(\frac{\Delta(\mathbf{x}-\mathbf{y}_1)}{\sqrt{g}}\right)$ is made of a special type of partie finie delta-pseudo-function associated with the regularization (following the definition given in [1]). It involves (minus) the determinant of the metric $g_{\rho\sigma}$, namely g, evaluated at the point $(t,\mathbf{x})$, and a generalization $\mathrm{Pf}\Delta(\mathbf{x}-\mathbf{y}_1)$ of the Dirac function defined in such a way that its action on a singular function yields the value of the function at the singular point in the sense of the regularization. Among the rules for handling the delta-pseudo-functions, we are allowed to write $\mathrm{Pf}\left(\frac{\Delta(\mathbf{x}-\mathbf{y}_1)}{\sqrt{g}}\right) = \frac{1}{\sqrt{g}}\mathrm{Pf}\Delta(\mathbf{x}-\mathbf{y}_1)$, whereas it is strictly forbidden to replace the latter quantity by $[\frac{1}{\sqrt{g}}]_1 \mathrm{Pf}\Delta(\mathbf{x}-\mathbf{y}_1)$.

The stress-energy tensor (1.1) takes the same form as the one of test particles moving in a fixed background, but with the role of the background played by the regularized metric generated by the bodies. In particular, the equations of motion obtained from the covariant conservation of that tensor ($\nabla_\nu T^{\mu\nu}_{\rm particle} = 0$), take the same form as the "geodesic equations", when considered with respect to the regularized metric. Our definition of the stress-energy tensor (1.1) constitutes a proposal, that we have found to be the most natural in the problem of the equations of binary motion at the 3PN order [24,25], but that we have not proved to be generally valid to higher post-Newtonian orders (nor of course when considered outside a framework of post-Newtonian expansions). The tensor (1.1) appears to be a good candidate for the characterization of point-like particles in post-Newtonian expansions of general relativity.

The plan of this paper is the following. In Section II, we recall from [1] the material needed in the subsequent parts concerning the Hadamard regularization and the associated pseudo-functions. In Section III, we investigate the formulas, needed to regularize, for the Lorentz transformation of some field point as well as two source points, and we define the new regularization around one of the particles as taking place within the instantaneous spatial hypersurface of the particle. In Section IV, we give the formulas for this regularization at the level of the first relativistic correction $1/c^2$. Finally, in Section V, we derive from an action principle our model of stress-energy tensor of point-like particles; the covariant conservation of this tensor leads to the equations of motion.

II. HADAMARD REGULARIZATION

To make the present paper self-contained, we shall review in this section the classic notions of the Hadamard regularization of singular functions and divergent integrals [2,3],

as well as the construction, by Blanchet and Faye [1], of a set of pseudo-functions associated with it. We follow closely the investigation of our previous paper [1] and employ most of its notation. A coordinate system $\{t, \mathbf{x}\}$ being given on space-time (for instance the harmonic coordinates used in Section IV), we consider some functions $F(\mathbf{x})$ defined on the spatial slice t =const, where $\mathbf{x} \in \mathbb{R}^3$ denotes the position in the slice. We say that the function $F(\mathbf{x})$ belongs to the class $\mathcal{F}$ if and only if F is a smooth function on $\mathbb{R}^3$ except at two isolated points $\mathbf{y}_1$ and $\mathbf{y}_2$, and admits around each of these points the following power-like singular expansions. Denoting by $r_1 = |\mathbf{x} - \mathbf{y}_1|$ the spatial distance to the point 1, and by $\mathbf{n}_1 = (\mathbf{x} - \mathbf{y}_1)/r_1$ the spatial direction of approach to 1, we assume that, for any $N \in \mathbb{N}$,

$$F(\mathbf{x}) = \sum_{a_0 \leq a \leq N} r_1^a \underset{1}{f}_a(\mathbf{n}_1) + o(r_1^N) \; . \tag{2.1}$$

The coefficients ${}_1f_a$ of the various powers of r_1 are smooth functions of the unit vector $\mathbf{n}_1$, and the remainder tends to zero strictly more rapidly than r_1^N when $r_1 \to 0$. The powers a of r_1 in that expansion are assumed to be real, $a \in \mathbb{R}$, to range in discrete steps, i.e. $a \in (a_i)_{i \in \mathbb{N}}$, and to be bounded from below, i.e. $a_0 \leq a$ for some $a_0 \in \mathbb{R}$. Similarly, we assume the same type of expansion around the point 2,

$$\forall N \in \mathbb{N} \; , \quad F(\mathbf{x}) = \sum_{b_0 \leq b \leq N} r_2^b \underset{2}{f}_b(\mathbf{n}_2) + o(r_2^N) \; , \tag{2.2}$$

where $r_2 = |\mathbf{x} - \mathbf{y}_2|$ and $\mathbf{n}_2 = (\mathbf{x} - \mathbf{y}_2)/r_2$. Thus, to each function F in the class $\mathcal{F}$ are associated two discrete families of indices a and b, and two corresponding families of coefficients ${}_1f_a(\mathbf{n}_1)$ and ${}_2f_b(\mathbf{n}_2)$, all of them depending on F. We shall refer to the coefficients ${}_1f_a$ for which $a < 0$ (and similarly to ${}_2f_b$ when $b < 0$) as the *singular* coefficients of F in the expansion when $r_1 \to 0$. Since $a \geq a_0(F)$ and $b \geq b_0(F)$, the number of singular coefficients of F is always finite.

The so-called "partie finie" in the sense of Hadamard [2,3] of the singular function F at the location of the singular point $\mathbf{y}_1$ is equal to the angular average, say ${}_1\hat{f}_0$, of the zeroth-order coefficient, ${}_1f_0(\mathbf{n}_1)$, in the expansion of the function when $r_1 \to 0$ we assumed in (2.1); namely

$$(F)_1 = \underset{1}{\hat{f}}_0 \equiv \int \frac{d\Omega_1}{4\pi} \underset{1}{f}_0(\mathbf{n}_1) \; , \tag{2.3}$$

where $d\Omega_1 = d\Omega(\mathbf{n}_1)$ denotes the solid angle element of origin $\mathbf{y}_1$ and direction $\mathbf{n}_1$; the latter angular integration is performed within the coordinate hypersurface t =const. A crucial property of the Hadamard partie finie is its "non-distributivity" with respect to the multiplication, in the sense that

$$(FG)_1 \neq (F)_1 (G)_1 \tag{2.4}$$

in general. When applied to the gradient $\partial_i F$ of a function $F \in \mathcal{F}$, the definition (2.3) yields a useful formula which permits one to compute rapidly the partie finie of complicated expressions involving gradients:

$$(\partial_i F)_1 = 3\Big(\frac{n_1^i}{r_1}F\Big)_1 \ . \tag{2.5}$$

Closely related to the concept of partie finie of a singular function is the definition of the partie finie (Pf) of the divergent integral $\int d^3\mathbf{x}\ F$. All-over this paper, we assume that the functions decrease fast enough at infinity (when $|\mathbf{x}| \to +\infty$) so that the possible divergencies of integrals come only from the bounds located at the two singular points 1 and 2. The "partie-finie integral" reads [2,3] as

$$\mathrm{Pf}_{s_1,s_2}\int d^3\mathbf{x}\ F = \lim_{s\to 0}\Bigg\{\int_{\mathbb{R}^3\setminus \mathrm{B}_1(s)\cup \mathrm{B}_2(s)} d^3\mathbf{x}\ F \\ + 4\pi \sum_{a+3<0} \frac{s^{a+3}}{a+3}\left(\frac{F}{r_1^a}\right)_1 + 4\pi \ln\left(\frac{s}{s_1}\right)\left(r_1^3 F\right)_1 + 1 \leftrightarrow 2\Bigg\} \ . \tag{2.6}$$

The integral in the right side extends over $\mathbb{R}^3$ deprived from two closed spherical balls $\mathrm{B}_1(s)$ and $\mathrm{B}_2(s)$ of radius s centered on the two singularities [thus $\mathrm{B}_1(s)$ and $\mathrm{B}_2(s)$ are defined by $r_1 \leq s$ and $r_2 \leq s$]. The other terms, which are defined by means of the partie finie in the sense of (2.3), are chosen in such a way that the limit $s \to 0$ exists. The notation $1 \leftrightarrow 2$ indicates the same terms as the two previous ones but corresponding to the other point. The summation index a satisfies $a_0 \leq a < -3$ (in particular the sum is always finite). Notice the two arbitrary constants s_1 and s_2 which are introduced in order to adimensionalize the arguments of the logarithms in (2.6); the partie finie owns an ambiguity through these constants (hence the notation Pf_{s_1,s_2}). The close connection between the partie finie of a singular function (2.3) and that of a divergent integral (2.6) is most easily seen from the fact that [1]

$$\mathrm{Pf}\int d^3\mathbf{x}\ \partial_i F = -4\pi (n_1^i r_1^2 F)_1 + 1 \leftrightarrow 2 \ . \tag{2.7}$$

Unlike in the case of continuous functions, the (partie-finie) integral of a gradient is non-zero in general, and equal to the sum of the parties finies, in the sense of (2.3), of the surface integrals surrounding the singularities, in the limit where the surface areas tend to zero. This fact motivated the introduction and study in [1] of a new derivative operator acting on $\mathcal{F}$, satisfying a property of "integration by parts" implying that the integral of any gradient is always zero. This operator generalizes for the class of functions $\mathcal{F}$ the standard distributional derivative of Schwartz [3].

Let us associate to any $F \in \mathcal{F}$ a pseudo-function denoted $\mathrm{Pf}F$ and defined to be the following linear form acting on the class $\mathcal{F}$:

$$\forall G \in \mathcal{F}\ , \quad <\mathrm{Pf}F, G> = \mathrm{Pf}\int d^3\mathbf{x}\ FG \ , \tag{2.8}$$

where the right side is a partie-finie integral in the sense of (2.6); we use a duality bracket to denote the result of the action of the pseudo-function $\mathrm{Pf}F$ on G. A fundamental definition adopted in [1], and motivated by the application to Physics, concerns the product of two

pseudo-functions, or of a function and a pseudo-function, which is the "*ordinary*" pointwise product in the sense that

$$\mathrm{Pf}F\,.\,\mathrm{Pf}G = F\,.\,\mathrm{Pf}G = G\,.\,\mathrm{Pf}F = \mathrm{Pf}(FG)\ . \tag{2.9}$$

Thus, for instance,

$$< \mathrm{Pf}F\,.\,\mathrm{Pf}G, H >= \mathrm{Pf}\int d^3\mathbf{x}\ FGH\ . \tag{2.10}$$

The product (2.9) chosen in [1] dictates most of the subsequent properties of the pseudo-functions, as well as their generalized distributional derivatives. (Refer to [27–29] for mathematical treatises on generalized functions and distributions.) In particular, the derivatives do not in general satisfy the Leibniz rule for the derivation of the product, although they satisfy it in an "integrated sense", according to the rule of integration by parts.

The Riesz [26] delta-function, given for $\varepsilon > 0$ by ${}_\varepsilon\delta(\mathbf{x}) = \frac{\varepsilon(1-\varepsilon)}{4\pi}\,|\mathbf{x}|^{\varepsilon-3}$, tends, in the usual sense of distribution theory, towards the Dirac measure when $\varepsilon \to 0$. When considered with respect to the singular point $\mathbf{y}_1$, the Riesz delta-function allows us to define a useful element of our class,

$${}_\varepsilon\delta_1(\mathbf{x}) \equiv {}_\varepsilon\delta(\mathbf{x}-\mathbf{y}_1) = \tfrac{\varepsilon(1-\varepsilon)}{4\pi}\, r_1^{\varepsilon-3}\ \in\ \mathcal{F}\ . \tag{2.11}$$

Therefore it is possible to associate to ${}_\varepsilon\delta_1$ (for any $\varepsilon > 0$) the pseudo-function $\mathrm{Pf}_\varepsilon\delta_1$ following the prescription (2.8). Applying the limit $\varepsilon \to 0$, we obtain [1]

$$\lim_{\varepsilon\to 0}\ < \mathrm{Pf}_\varepsilon\delta_1, F >\equiv \lim_{\varepsilon\to 0}\ \mathrm{Pf}\int d^3\mathbf{x}\ {}_\varepsilon\delta_1 F = (F)_1\ , \tag{2.12}$$

where the value of F at the point 1 in the right side is defined by the prescription (2.3). This motivates us for introducing a new pseudo-function, we shall call the delta-pseudo-function $\mathrm{Pf}\delta_1$, as the formal limit of the pseudo-functions $\mathrm{Pf}_\varepsilon\delta_1$ when $\varepsilon \to 0$. By definition,

$$\forall F\in\mathcal{F},\quad < \mathrm{Pf}\delta_1, F >= (F)_1\ . \tag{2.13}$$

Clearly, the delta-pseudo-function $\mathrm{Pf}\delta_1$ generalizes the notion of Dirac distribution $\delta_1 \equiv \delta(\mathbf{x}-\mathbf{y}_1)$ to the case where the "test" functions are singular and belong to the class $\mathcal{F}$. Extending the definition of the product (2.9) to include the delta-pseudo-function we pose

$$\mathrm{Pf}F\,.\,\mathrm{Pf}\delta_1 = F\,.\,\mathrm{Pf}\delta_1 = \mathrm{Pf}(F\delta_1)\ , \tag{2.14}$$

as well as, for instance,

$$\mathrm{Pf}(F\delta_1)\,.\,\mathrm{Pf}G = \mathrm{Pf}(F\delta_1)\,.\,G = \mathrm{Pf}(FG\delta_1)\ . \tag{2.15}$$

The new object $\mathrm{Pf}(F\delta_1)$ in (2.14)-(2.15) has no equivalent in distribution theory; it satisfies

$$\forall G\in\mathcal{F}\ ,\quad < \mathrm{Pf}(F\delta_1), G >= (FG)_1\ . \tag{2.16}$$

We notice for future reference that a consequence of the "non-distributivity" of the Hadamard partie finie [see (2.4)] is that

$$\mathrm{Pf}(F\delta_1) \neq (F)_1 \; \mathrm{Pf}\delta_1 \; . \tag{2.17}$$

We are not allowed to replace a singular function that appears in factor of the delta-pseudo-function at the point 1 by its regularized value at that point.

The derivative of the delta-pseudo-function $\mathrm{Pf}\delta_1$ was constructed in [1]. As it turns out, it takes the form of an "ordinary" derivative : $\partial_i(\mathrm{Pf}\delta_1) = \mathrm{Pf}(\partial_i\delta_1)$; due to the presence of the delta-pseudo-function, there are no distributional terms associated with it. We have simply (from the rule of integration by parts),

$$\forall F \in \mathcal{F} \; , \qquad < \partial_i(\mathrm{Pf}\delta_1), F >= - < \mathrm{Pf}\delta_1, \partial_i F >= -(\partial_i F)_1 \; . \tag{2.18}$$

The differentiation of the more complicated object $\mathrm{Pf}(F\delta_1)$ proceeds in the same way:

$$\forall G \in \mathcal{F} \; , \qquad < \partial_i[\mathrm{Pf}(F\delta_1)], G >= - < \mathrm{Pf}(F\delta_1), \partial_i G >= -(F\partial_i G)_1 \; . \tag{2.19}$$

Note that, as a consequence of the identity (2.5), we can write for the intrinsic form of this object:

$$\partial_i[\mathrm{Pf}(F\delta_1)] = \mathrm{Pf}\left[r_1^3 \partial_i \left(\frac{F}{r_1^3} \right) \delta_1 \right] \; . \tag{2.20}$$

Because the derivative of the delta-pseudo-function is equal to the ordinary one, the Leibniz rule for the derivative of a product happens to still hold. For instance, in the case of the product of $\mathrm{Pf}(F\delta_1)$ with some pseudo-function $\mathrm{Pf}G$, we have

$$\partial_i[\mathrm{Pf}(F\delta_1).\,\mathrm{Pf}G] = \partial_i[\mathrm{Pf}(F\delta_1)].\,\mathrm{Pf}G + \mathrm{Pf}(F\delta_1).\,\partial_i(\mathrm{Pf}G) \; . \tag{2.21}$$

The proof uses the combination of (2.15) and (2.19).

III. LORENTZIAN REGULARIZATION

To define a Lorentzian regularization *à la* Hadamard (based on the investigation of [1] and on Section II), we now need to specify in a precise way the dependence of a function $F(\mathbf{x})$ in the class $\mathcal{F}$ on the "source" variables at the coordinate time t of a global frame $\{\mathbf{x}, t\}$. We assume (as everywhere else in this paper) that we are working at some given finite order in a relativistic or post-Newtonian approximation. Up to a given order, we can choose as the source variables the two trajectories $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ in the frame $\{\mathbf{x}, t\}$, and the two coordinate velocities $\mathbf{v}_1(t) = d\mathbf{y}_1/dt$ and $\mathbf{v}_2(t) = d\mathbf{y}_2/dt$ (the trajectories of the particles are time-like world lines in Minkowski space-time). Indeed, it is legitimate to assume only the latter source variables because, up to a given post-Newtonian order, we can order-reduce the accelerations and all derivatives of accelerations by means of the equations of motion of the

particles up to the appropriate accuracy (in general the precision of the equations of motion needed for this order-reduction is one order less than the given post-Newtonian order at which we are performing a calculation). Of course, we are assuming that these equations of motion are known (they are known presently to the 2.5PN order [13,14,19], and the general motivation of this work is to get them up to the 3PN order [22,24,25]. Thus, we assume that the function $F \in \mathcal{F}$ really reads

$$F(\mathbf{x},t) = F[\mathbf{x}; \mathbf{y}_1(t), \mathbf{y}_2(t); \mathbf{v}_1(t), \mathbf{v}_2(t)] \; . \tag{3.1}$$

We denote with the same letter F, by a slight abuse of notation, the function of the field point $(\mathbf{x},t)$ and the functional of the field point and source variables in the right-hand-side. For definiteness, we assume that the two trajectories are smooth functions of time, i.e. $\mathbf{y}_1$, $\mathbf{y}_2 \in C^\infty(\mathbb{R}^3)$, and that F is a smooth functional of the two velocities $\mathbf{v}_1$, $\mathbf{v}_2$ (see also Section IX of [1] for details about our assumptions). By (3.1), we mean that the dependence of F on the coordinate time t is through (and only through) the two instantaneous trajectories $\mathbf{y}_1$, $\mathbf{y}_2$ and velocities $\mathbf{v}_1$, $\mathbf{v}_2$. Note also that it is implicitely assumed with our notation (3.1) that the function F depends *locally* on time t (no dependence over the trajectories and velocities at some time earlier than t for instance). Furthermore, very often in applications, we shall find that the dependence of F on the spatial position $\mathbf{x}$ appears only via the two spatial distances to the source points, $\mathbf{r}_1(t) = \mathbf{x} - \mathbf{y}_1(t)$ and $\mathbf{r}_2(t) = \mathbf{x} - \mathbf{y}_2(t)$. In this paper, we shall generally suppose, in order to simplify the presentation, that this is the case; namely, the function F, as a functional of the source variables, is

$$F(\mathbf{x},t) = F[\mathbf{r}_1(t), \mathbf{r}_2(t); \mathbf{v}_1(t), \mathbf{v}_2(t)] \; . \tag{3.2}$$

The hypothesis (3.2) does not constitute a very severe restriction. The extension to the more general case (3.1) is generally straightforward; moreover, (3.2) is always verified in the problem of the post-Newtonian equations of motion of binary systems. In this section, we shall define the Lorentzian regularized value of the function F at the location of the singularity 1, by contrast to the non-invariant regularized value defined by (2.3) within the "global" coordinate hypersurface $t =$const. We shall denote by $[F]_1$ the new Lorentzian regularization of F at the point 1, defined within the instantaneous rest frame of the particle 1 at $t' =$const [in constrast with the notation $(F)_1$ used in (2.3) for the old regularization]. In addition, we shall introduce a delta-pseudo-function denoted by $\mathrm{Pf}\Delta_1$ associated with the new regularization [similarly to the delta-pseudo-function $\mathrm{Pf}\delta_1$ which was defined in (2.13) in the case of the old regularization].

A. Lorentz transformation of the source variables

In this paper, it is sufficient to consider only those homogeneous proper Lorentz transformations which change the velocity of a global inertial frame $\{x^\mu\} = \{ct, \mathbf{x}\}$. More specifically, let us consider the Lorentz boost

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}(\mathbf{V})\, x^{\nu}\ , \tag{3.3}$$

where the Lorentz matrix $\Lambda^{\mu}{}_{\nu}(\mathbf{V})$, depending on the constant boost velocity $\mathbf{V}$, is given by

$$\Lambda^{0}{}_{0}(\mathbf{V}) = \gamma\ , \tag{3.4a}$$
$$\Lambda^{i}{}_{0}(\mathbf{V}) = -\gamma\frac{V^i}{c}\ , \tag{3.4b}$$
$$\Lambda^{0}{}_{j}(\mathbf{V}) = -\gamma\frac{V_j}{c}\ , \tag{3.4c}$$
$$\Lambda^{i}{}_{j}(\mathbf{V}) = \delta^i_j + \frac{\gamma^2}{\gamma+1}\frac{V^iV_j}{c^2}\ . \tag{3.4d}$$

We indifferently denote the components of the boost vector by $\mathbf{V} = (V^i) = (V_i)$ (spatial indices $i, j = 1, 2, 3$). The Lorentz factor γ reads

$$\gamma = \frac{1}{\sqrt{1-\frac{\mathbf{V}^2}{c^2}}}\ , \tag{3.5}$$

with $\mathbf{V}^2 = \delta_{ij}V^iV^j$ (of course $|\mathbf{V}| < c$). The inverse transformation is $x^{\nu} = \Lambda_{\mu}{}^{\nu}(\mathbf{V})\, x'^{\mu}$ where the components of $\Lambda_{\mu}{}^{\nu}(\mathbf{V}) = \eta_{\mu\rho}\eta^{\nu\sigma}\Lambda^{\rho}{}_{\sigma}(\mathbf{V})$ are obtained directly from (3.3) by changing $\mathbf{V} \to -\mathbf{V}$. The choice of sign made in the $0i$ components of the boost (3.4) is such that a particle which has velocity $\mathbf{V}$ at time t in the frame $\{x^{\mu}\}$ is at rest in the frame $\{x'^{\mu}\}$ at time t'.

We introduce on one side the space-time event Q, which represents for us a "field" point located outside the two world lines of the particles, and on the other side the space-time events P_1, M_1 and P_2, M_2, which are "source" points, lying respectively on the world lines of the particles 1 and 2 (see below for their definition). The coordinates of the event Q are $(t, \mathbf{x})$ in the frame $\{x^{\mu}\}$ and $(t', \mathbf{x}')$ in the frame $\{x'^{\mu}\}$. Sorting out the spatial and temporal indices in (3.3), we have

$$ct' = c\Lambda^{0}{}_{0}t + \Lambda^{0}{}_{j}x^j\ , \tag{3.6a}$$
$$x'^i = c\Lambda^{i}{}_{0}t + \Lambda^{i}{}_{j}x^j\ . \tag{3.6b}$$

The points P_1 and P_2 are now defined as the two events that are located on the trajectories of the particles and are "simultaneous" with the event Q in the frame $\{x^{\mu}\}$, i.e. that belong to the same spatial slice t =const as Q. The coordinates of P_1 and P_2 in $\{x^{\mu}\}$ are denoted by $(t, \mathbf{y}_1)$ and $(t, \mathbf{y}_2)$ respectively, the two trajectories $\mathbf{y}_1 = \mathbf{y}_1(t)$ and $\mathbf{y}_2 = \mathbf{y}_2(t)$ being parametrized by the coordinate time t in that frame. On the other hand, in the new frame $\{x'^{\mu}\}$, the coordinates of P_1 and P_2 are $(\tau'_1, \mathbf{z}'_1)$ and $(\tau'_2, \mathbf{z}'_2)$. Evidently, the primed coordinates are related to the unprimed ones by the Lorentz boost (3.3), so that

$$c\tau'_1 = c\Lambda^{0}{}_{0}t + \Lambda^{0}{}_{j}y_1^j\ , \tag{3.7a}$$
$$z'^{\,i}_1 = c\Lambda^{i}{}_{0}t + \Lambda^{i}{}_{j}y_1^j\ , \tag{3.7b}$$

in the case of the event P_1 [where $y_1^j = y_1^j(t)$, $y_2^j = y_2^j(t)$], and

$$c\tau'_2 = c\Lambda^0{}_0 t + \Lambda^0{}_j y_2^j \ , \tag{3.8a}$$

$$z'^{\,i}_2 = c\Lambda^i{}_0 t + \Lambda^i{}_j y_2^j \ , \tag{3.8b}$$

in the case of the event P_2. In the new frame $\{x'^\mu\}$, the source events that are simultaneous with Q are not P_1 and P_2, but some other events M_1 and M_2, whose coordinates in the primed frame are thus $(t', \mathbf{y}'_1)$ and $(t', \mathbf{y}'_2)$; the coordinate time t' is the same as that of Q in the primed frame, and the spatial coordinates are the trajectories of the particles $\mathbf{y}'_1 = \mathbf{y}'_1(t')$ and $\mathbf{y}'_2 = \mathbf{y}'_2(t')$ which are labelled by t' in the new frame. Let $(\tau_1, \mathbf{z}_1)$ and $(\tau_2, \mathbf{z}_2)$ be the coordinates of M_1 and M_2 in the original frame $\{x^\mu\}$. By definition,

$$ct' = c\Lambda^0{}_0 \tau_1 + \Lambda^0{}_j z_1^j \ , \tag{3.9a}$$

$$y'^{\,i}_1 = c\Lambda^i{}_0 \tau_1 + \Lambda^i{}_j z_1^j \ , \tag{3.9b}$$

$$ct' = c\Lambda^0{}_0 \tau_2 + \Lambda^0{}_j z_2^j \ , \tag{3.9c}$$

$$y'^{\,i}_2 = c\Lambda^i{}_0 \tau_2 + \Lambda^i{}_j z_2^j \ , \tag{3.9d}$$

where $y'^{\,i}_1 = y'^{\,i}_1(t')$ and $y'^{\,i}_2 = y'^{\,i}_2(t')$. Beware of our notation, where τ'_1 (for instance) is the time coordinate of P_1 in $\{x'^\mu\}$ while τ_1 is the time coordinate in $\{x^\mu\}$ of the *different* event M_1. Since the events M_1 and M_2 are located on the world lines of the particles parametrized by $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ in $\{x^\mu\}$, it is clear that at time τ_1 in that frame their coordinates are related to the trajectories by

$$\mathbf{z}_1 = \mathbf{y}_1(\tau_1) \ , \tag{3.10a}$$

$$\mathbf{z}_2 = \mathbf{y}_2(\tau_2) \ . \tag{3.10b}$$

Similarly, from the fact that P_1 and P_2 are also on the world lines, which write as $\mathbf{y}'_1(t')$ and $\mathbf{y}'_2(t')$ in the frame $\{x'^\mu\}$, we deduce that their coordinates in $\{x'^\mu\}$ satisfy

$$\mathbf{z}'_1 = \mathbf{y}'_1(\tau'_1) \ , \tag{3.11a}$$

$$\mathbf{z}'_2 = \mathbf{y}'_2(\tau'_2) \ . \tag{3.11b}$$

By eliminating t' from the equations (3.6a) and (3.9a) we immediately obtain

$$c\Lambda^0{}_0(\tau_1 - t) = \Lambda^0{}_i(x^i - z_1^i) \ , \tag{3.12}$$

or, equivalently, taking also into account (3.4),

$$\tau_1 - t = -\frac{1}{c^2}\mathbf{V}.(\mathbf{x} - \mathbf{z}_1) \ , \tag{3.13}$$

where the usual Euclidean scalar product between (boldface) vectors is denoted by a dot. With the help of the latter formula for expressing τ_1, we can re-state the belonging of $\mathbf{z}_1$ to the particle world line at time τ_1 [see (3.10a)] as

$$\mathbf{z}_1 = \mathbf{y}_1\left(t - \frac{1}{c^2}\mathbf{V}.(\mathbf{x}-\mathbf{z}_1)\right) . \tag{3.14}$$

Recall that $\mathbf{z}_1$ is the spatial coordinate in the old frame of the event M_1 which is simultaneous with the field point Q in the *new* frame. Clearly, the equation (3.14) determines the vector $\mathbf{z}_1$ as a function of the coordinates $(t, \mathbf{x})$ of the field-point event Q (see the appendix). Here, let us view $\mathbf{z}_1$ as a "vector" field $\mathbf{z}_1(\mathbf{x})$, solution of (3.14), lying in the three-dimensional space t =const. It is evident from (3.14) that the function $\mathbf{z}_1(\mathbf{x})$ admits a fixed point at $\mathbf{y}_1 = \mathbf{y}_1(t)$, in the sense that

$$\mathbf{z}_1(\mathbf{y}_1) = \mathbf{y}_1 . \tag{3.15}$$

Unless specified otherwise [like in (3.14)], the notation $\mathbf{y}_1$ always means $\mathbf{y}_1(t)$. The mathematical justification of (3.15) is the following. From the fact that the world line of the particle is time-like we can write, for any instants $\hat{t}$ and $\bar{t}$, the inequality $|\mathbf{y}_1(\hat{t}) - \mathbf{y}_1(\bar{t})| < c|\hat{t} - \bar{t}|$. Hence, applying the definition (3.14), we find that our function $\mathbf{z}_1(\mathbf{x})$ obeys, for any positions $\hat{\mathbf{x}}$ and $\bar{\mathbf{x}}$, the further inequalities $|\mathbf{z}_1(\hat{\mathbf{x}}) - \mathbf{z}_1(\bar{\mathbf{x}})| < \frac{1}{c}|\mathbf{V}.(\hat{\mathbf{x}} - \bar{\mathbf{x}})| \leq \frac{|\mathbf{V}|}{c}|\hat{\mathbf{x}} - \bar{\mathbf{x}}|$. Now recall that $\frac{|\mathbf{V}|}{c} < 1$, so the latter inequalities mean exactly that the function $\mathbf{x} \to \mathbf{z}_1(\mathbf{x})$ is a *contracting* application with respect to the usual Euclidean norm (i.e., it satisfies the property of Lipschitz with a ratio $k = \frac{|\mathbf{V}|}{c}$ strictly less than one). Therefore, by the theorem of Picard, the function admits a unique fixed point, which of course is nothing but $\mathbf{y}_1$. (Besides, at the location of the fixed point, we have $\tau_1 = t$.)

In this paper, we establish the general solution of the equation (3.14) in the form of an infinite (post-Newtonian) power series in $1/c^2$. We shall not discuss the convergence properties of this series and simply employ it to define the regularization up to any relativistic order. This is sufficient for the application to the problem of the equations of motion of particles in the post-Newtonian approximation. The general solution of (3.14), as determined in the appendix, reads

$$\mathbf{z}_1 = \mathbf{y}_1 + \sum_{n=1}^{+\infty} \frac{(-)^n}{c^{2n} n!} \left(\frac{\partial}{\partial t}\right)^{n-1} \Big[(\mathbf{V}.\mathbf{r}_1)^n \mathbf{v}_1\Big] , \tag{3.16}$$

with shorthand notations $\mathbf{y}_1 = \mathbf{y}_1(t)$, $\mathbf{r}_1 = \mathbf{x} - \mathbf{y}_1(t)$ and $\mathbf{v}_1 = \mathbf{v}_1(t)$. The many derivatives $\partial/\partial t$ in the right side are partial time derivatives with respect to the coordinate time t, the spatial coordinate $\mathbf{x}$ being held constant. They act on $\mathbf{r}_1$ through the trajectory $\mathbf{y}_1$: we have $\partial \mathbf{r}_1/\partial t = -\mathbf{v}_1$ or $\partial(\mathbf{V}.\mathbf{r}_1)/\partial t = -\mathbf{V}.\mathbf{v}_1$ for instance. On the other side, they act of course on velocities and (derivatives of) accelerations: thus $\partial \mathbf{v}_1/\partial t = \mathbf{a}_1$, $\partial \mathbf{a}_1/\partial t = \mathbf{b}_1$, $\partial \mathbf{b}_1/\partial t = \mathbf{c}_1$, and so on, where $\mathbf{a}_1$, $\mathbf{b}_1$, $\mathbf{c}_1$ represent the acceleration, and its first and second derivatives (in these cases the partial derivative is a total derivative, e.g. $d\mathbf{v}_1/dt = \mathbf{a}_1$). Thus, to high post-Newtonian order, (3.16) contains many accelerations and derivatives of accelerations, but it is understood that this formula is order-reduced, consistently with the post-Newtonian order; i.e. all accelerations and derivatives of accelerations are to be replaced by the functionals of the positions and velocities deduced from the equations of

motion. Combining (3.13) and (3.16), we easily find the corresponding solution for the time coordinate τ_1,

$$\tau_1 = t + \sum_{n=1}^{+\infty} \frac{(-)^n}{c^{2n} n!} \left(\frac{\partial}{\partial t}\right)^{n-1} \Big[(\mathbf{V}.\mathbf{r}_1)^n\Big] \; . \tag{3.17}$$

[Of course, since $\mathbf{V}$ is a constant vector, it could be as well put outside the partial time derivative operators in both (3.16) and (3.17).] Finally, equations (3.16) and (3.17) determine completely the space-time event M_1. From them, we can recover directly the fact that when $\mathbf{x} = \mathbf{y}_1$ (at the fixed point) then $\mathbf{z}_1 = \mathbf{y}_1$ and $\tau_1 = t$: there are in the right sides of both relations $n-1$ partial time derivatives acting on a term that involves the nth power $(\mathbf{V}.\mathbf{r}_1)^n$, so that at least one of the scalar products $\mathbf{V}.\mathbf{r}_1$ is left un-differentiated, and makes the sums in (3.16)-(3.17) vanish when $\mathbf{r}_1 = 0$. Replacing both $\mathbf{z}_1$ and τ_1 as given by the infinite post-Newtonian series back into the relation (3.10a), expressing both sides of the resulting equation as the same type of post-Newtonian series with the help of a formal Taylor expansion when $c \to \infty$, and finally equating all the coefficients of these two series, yields an interesting mathematical formula relating together some sums of products of derivatives. This formula is derived in the appendix (where we present also a direct proof of it). Notice that the same reasoning as before can be done on the coordinates of the event P_1 in the new frame, that we find to be given by

$$\mathbf{z}'_1 = \mathbf{y}'_1 + \sum_{n=1}^{+\infty} \frac{1}{c^{2n} n!} \left(\frac{\partial}{\partial t'}\right)^{n-1} \Big[(\mathbf{V}.\mathbf{r}'_1)^n \mathbf{v}'_1\Big] \; , \tag{3.18a}$$

$$\tau'_1 = t' + \sum_{n=1}^{+\infty} \frac{1}{c^{2n} n!} \left(\frac{\partial}{\partial t'}\right)^{n-1} \Big[(\mathbf{V}.\mathbf{r}'_1)^n\Big] \; , \tag{3.18b}$$

where $\mathbf{y}'_1 = \mathbf{y}'_1(t')$, $\mathbf{r}'_1 = \mathbf{x}' - \mathbf{y}'_1(t')$ and $\mathbf{v}'_1 = \mathbf{v}'_1(t')$. Evidently, the result (3.18) can also be deduced directly from (3.16)-(3.17) by changing $\mathbf{V}$ into $-\mathbf{V}$ and replacing all the non-primed variables by the corresponding primed ones.

We are now able to give all the transformation laws of field and source variables between the frames $\{x^\mu\}$ and $\{x'^\mu\}$. Of course, from (3.6), the transformation of the field variables is the standard Lorentz one,

$$t' = \gamma \left(t - \frac{1}{c^2}(\mathbf{V}.\mathbf{x}) \right) \; , \tag{3.19a}$$

$$\mathbf{x}' = \mathbf{x} - \gamma \mathbf{V} \left(t - \frac{1}{c^2}\frac{\gamma}{\gamma+1}(\mathbf{V}.\mathbf{x}) \right) \; . \tag{3.19b}$$

Concerning the source variables, we are interested in the expressions of the new positions $\mathbf{y}'_1(t')$, $\mathbf{y}'_2(t')$ and velocities $\mathbf{v}'_1(t')$, $\mathbf{v}'_2(t')$ in the new frame at time t'. These are straightforwardly obtained from inserting the results (3.16) and (3.17) into the equations (3.9), as well as the similar results corresponding to the point 2. We find, for trajectories,

$$\mathbf{y}'_1 = \mathbf{y}_1 - \gamma\mathbf{V}\left(t - \frac{1}{c^2}\frac{\gamma}{\gamma+1}(\mathbf{V}.\mathbf{x})\right) + \sum_{n=1}^{+\infty}\frac{(-)^n}{c^{2n}n!}\left(\frac{\partial}{\partial t}\right)^{n-1}\left[(\mathbf{V}.\mathbf{r}_1)^n\left(\mathbf{v}_1 - \frac{\gamma}{\gamma+1}\mathbf{V}\right)\right] , \tag{3.20a}$$

$$\mathbf{y}'_2 = \mathbf{y}_2 - \gamma\mathbf{V}\left(t - \frac{1}{c^2}\frac{\gamma}{\gamma+1}(\mathbf{V}.\mathbf{x})\right) + \sum_{n=1}^{+\infty}\frac{(-)^n}{c^{2n}n!}\left(\frac{\partial}{\partial t}\right)^{n-1}\left[(\mathbf{V}.\mathbf{r}_2)^n\left(\mathbf{v}_2 - \frac{\gamma}{\gamma+1}\mathbf{V}\right)\right] . \tag{3.20b}$$

By subtracting the latter equations (3.20) to $\mathbf{x}'$ as given by (3.19b) we obtain the spatial distances $\mathbf{r}'_1 = \mathbf{x}' - \mathbf{y}'_1(t')$ and $\mathbf{r}'_2 = \mathbf{x}' - \mathbf{y}'_2(t')$ as

$$\mathbf{r}'_1 = \mathbf{r}_1 - \sum_{n=1}^{+\infty}\frac{(-)^n}{c^{2n}n!}\left(\frac{\partial}{\partial t}\right)^{n-1}\left[(\mathbf{V}.\mathbf{r}_1)^n\left(\mathbf{v}_1 - \frac{\gamma}{\gamma+1}\mathbf{V}\right)\right] , \tag{3.21a}$$

$$\mathbf{r}'_2 = \mathbf{r}_2 - \sum_{n=1}^{+\infty}\frac{(-)^n}{c^{2n}n!}\left(\frac{\partial}{\partial t}\right)^{n-1}\left[(\mathbf{V}.\mathbf{r}_2)^n\left(\mathbf{v}_2 - \frac{\gamma}{\gamma+1}\mathbf{V}\right)\right] . \tag{3.21b}$$

These relations will play the crucial role in the definition of our Lorentzian regularization. Of interest also is the expression of the relative distance between the two particles, i.e. $\mathbf{y}'_{12} = \mathbf{y}'_1 - \mathbf{y}'_2 = \mathbf{r}'_2 - \mathbf{r}'_1$ given by

$$\mathbf{y}'_{12} = \mathbf{y}_{12} + \sum_{n=1}^{+\infty}\frac{(-)^n}{c^{2n}n!}\left(\frac{\partial}{\partial t}\right)^{n-1}\left[(\mathbf{V}.\mathbf{r}_1)^n\left(\mathbf{v}_1 - \frac{\gamma}{\gamma+1}\mathbf{V}\right) - (\mathbf{V}.\mathbf{r}_2)^n\left(\mathbf{v}_2 - \frac{\gamma}{\gamma+1}\mathbf{V}\right)\right] . \tag{3.22}$$

Finally, we compute the expressions of the coordinate velocities $\mathbf{v}'_1(t') = d\mathbf{y}'_1/dt'$ and $\mathbf{v}'_2(t') = d\mathbf{y}'_2/dt'$ in the new frame. They follow immediately from the law of transformation of the time derivative : $\partial'_t = \gamma\partial_t + \gamma V^i\partial_i$, and we obtain

$$\mathbf{v}'_1 = \frac{1}{\gamma}\mathbf{v}_1 - \mathbf{V} + \frac{1}{\gamma}\sum_{n=1}^{+\infty}\frac{(-)^n}{c^{2n}n!}\left(\frac{\partial}{\partial t}\right)^{n}\left[(\mathbf{V}.\mathbf{r}_1)^n\left(\mathbf{v}_1 - \frac{\gamma}{\gamma+1}\mathbf{V}\right)\right] , \tag{3.23a}$$

$$\mathbf{v}'_2 = \frac{1}{\gamma}\mathbf{v}_2 - \mathbf{V} + \frac{1}{\gamma}\sum_{n=1}^{+\infty}\frac{(-)^n}{c^{2n}n!}\left(\frac{\partial}{\partial t}\right)^{n}\left[(\mathbf{V}.\mathbf{r}_2)^n\left(\mathbf{v}_2 - \frac{\gamma}{\gamma+1}\mathbf{V}\right)\right] . \tag{3.23b}$$

Notice that although the velocities $\mathbf{v}'_1(t')$ and $\mathbf{v}'_2(t')$ are some mere functions of the coordinate time t' in the new frame, they depend, when expressed in terms of quantities belonging to the old frame, on both time *and* space coordinates t and $\mathbf{x}$. This is obvious because by changing the space coordinate $\mathbf{x}$ of the field point Q while keeping t =const we change the time coordinate t' of the source events M_1 and M_2 and therefore the values of their particle velocities (soon as the trajectories are accelerated). This fact is important and has to be taken correctly into account in the regularization process defined in the next subsection.

The inverse formulas are obtained in the same way by substituting (3.18) into the inverse of (3.7). They correspond of course to changing $\mathbf{V}$ into $-\mathbf{V}$ and replacing everywhere the un-primed labels by primed ones. We find, for the spatial distances and velocities,

$$\mathbf{r}_1 = \mathbf{r}'_1 - \sum_{n=1}^{+\infty} \frac{1}{c^{2n} n!} \left(\frac{\partial}{\partial t'}\right)^{n-1} \left[(\mathbf{V}.\mathbf{r}'_1)^n \left(\mathbf{v}'_1 + \frac{\gamma}{\gamma+1}\mathbf{V} \right) \right] , \tag{3.24a}$$

$$\mathbf{r}_2 = \mathbf{r}'_2 - \sum_{n=1}^{+\infty} \frac{1}{c^{2n} n!} \left(\frac{\partial}{\partial t'}\right)^{n-1} \left[(\mathbf{V}.\mathbf{r}'_2)^n \left(\mathbf{v}'_2 + \frac{\gamma}{\gamma+1}\mathbf{V} \right) \right] , \tag{3.24b}$$

$$\mathbf{v}_1 = \frac{1}{\gamma}\mathbf{v}'_1 + \mathbf{V} + \frac{1}{\gamma}\sum_{n=1}^{+\infty} \frac{1}{c^{2n} n!} \left(\frac{\partial}{\partial t'}\right)^{n} \left[(\mathbf{V}.\mathbf{r}'_1)^n \left(\mathbf{v}'_1 + \frac{\gamma}{\gamma+1}\mathbf{V} \right) \right] , \tag{3.24c}$$

$$\mathbf{v}_2 = \frac{1}{\gamma}\mathbf{v}'_2 + \mathbf{V} + \frac{1}{\gamma}\sum_{n=1}^{+\infty} \frac{1}{c^{2n} n!} \left(\frac{\partial}{\partial t'}\right)^{n} \left[(\mathbf{V}.\mathbf{r}'_2)^n \left(\mathbf{v}'_2 + \frac{\gamma}{\gamma+1}\mathbf{V} \right) \right] . \tag{3.24d}$$

B. Definition of the regularization

Let us consider a function F belonging to the class $\mathcal{F}$ and being at the same time a *scalar* under Lorentz transformations, i.e. $F(\mathbf{x},t) = F'(\mathbf{x}',t')$. More precisely, we restrict ourselves to the case of a dependence on $\mathbf{x}$ only via the distances $\mathbf{r}_1$ and $\mathbf{r}_2$ [*cf* (3.2)]; this means

$$F[\mathbf{r}_1(t), \mathbf{r}_2(t); \mathbf{v}_1(t), \mathbf{v}_2(t)] = F'[\mathbf{r}'_1(t'), \mathbf{r}'_2(t'); \mathbf{v}'_1(t'), \mathbf{v}'_2(t'); \mathbf{V}] , \tag{3.25}$$

where we use the same slighly abusive notation as in (3.2), with addition, in the right side, of the explicit mention of the dependence over the boost vector $\mathbf{V}$. All the variables in both frames $\{x^\mu\}$ and $\{x'^\mu\}$ are related to each other by the formulas developed in the previous subsection. The regularization process goes as follows.

(I) Starting from $F[\mathbf{r}_1, \mathbf{r}_2; \mathbf{v}_1, \mathbf{v}_2]$ defined in the frame $\{x^\mu\}$, we first determine the new functional $F'[\mathbf{r}'_1, \mathbf{r}'_2; \mathbf{v}'_1, \mathbf{v}'_2; \mathbf{V}]$ in the boosted frame $\{x'^\mu\}$. To do so, we replace all the variables $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{v}_1$, $\mathbf{v}_2$ by their expressions in terms of the new ones $\mathbf{r}'_1$, $\mathbf{r}'_2$, $\mathbf{v}'_1$, $\mathbf{v}'_2$ as given by the formulas (3.24), in which it is understood that all the accelerations are order-reduced up to some given specified post-Newtonian order. Performing all the necessary post-Newtonian re-expansions to that order, we indeed obtain in that way (since F is a Lorentz scalar) the new functional F' of the new distances $\mathbf{r}'_1$, $\mathbf{r}'_2$ and velocities $\mathbf{v}'_1$, $\mathbf{v}'_2$. In addition, F' depends as expected on the constant $\mathbf{V}$ which is yet un-specified at this stage.

(II) We compute the Hadamard regularization of F' at the point 1 following exactly the same rules as defined in (2.3), but in the boosted frame $\{x'^\mu\}$ (in particular, within the coordinate slice t' =const). In words, we perform the expansion of F' when the spatial distance r'_1 tends to zero, and obtain the same type of power-law expansion as in (2.1) [since the form

of the relations (3.24) shows that the structure of the expansions in both frames must be the same]. However, we get some primed functional coefficients ${}_1f'_a$ that differ from the un-primed coefficients ${}_1f_a$ appearing in (2.1). The boost vector $\mathbf{V}$ is simply held constant in the process. Thus, $\forall N \in \mathbb{N}$,

$$F'[\mathbf{r}'_1, \mathbf{r}'_2; \mathbf{v}'_1, \mathbf{v}'_2; \mathbf{V}] = \sum_{a_0 \leq a \leq N} {r'}_1^a \underset{1}{f'_a}\Big(\mathbf{n}'_1; \mathbf{y}'_{12}; \mathbf{v}'_1, \mathbf{v}'_2; \mathbf{V}\Big) + o\left({r'}_1^N\right) , \tag{3.26}$$

with the notation $r'_1 = |\mathbf{x}' - \mathbf{y}'_1|$, $\mathbf{n}'_1 = (\mathbf{x}' - \mathbf{y}'_1)/r'_1$ and $\mathbf{y}'_{12} = \mathbf{y}'_1 - \mathbf{y}'_2$. (The fact that the coefficients ${}_1f'_a$ depend on $\mathbf{y}'_{12}$ instead of the two individual trajectories $\mathbf{y}'_1$, $\mathbf{y}'_2$ is due to our restriction that F' depends on $\mathbf{x}'$ via the distances $\mathbf{r}'_1$ and $\mathbf{r}'_2$; also, the accelerations depend on the relative distance $\mathbf{y}'_{12}$.) Now, like in (2.3), we pick up the zeroth-order coefficient in the r'_1-expansion (3.26) and average over the angles. This defines a certain functional of the separation vector $\mathbf{y}'_{12}$, the velocities $\mathbf{v}'_1$, $\mathbf{v}'_2$ and the boost velocity $\mathbf{V}$,

$$\underset{1}{\hat{f}'_0}\Big(\mathbf{y}'_{12}; \mathbf{v}'_1, \mathbf{v}'_2; \mathbf{V}\Big) = \int \frac{d\Omega'_1}{4\pi} \underset{1}{f'_0}\Big(\mathbf{n}'_1; \mathbf{y}'_{12}; \mathbf{v}'_1, \mathbf{v}'_2; \mathbf{V}\Big) . \tag{3.27}$$

We insist that the angular average is performed *in the new frame*, within the spatial hypersurface t' =const; in particular, the solid angle element in (3.27) is the one associated with the unit direction $\mathbf{n}'_1$ in that hypersurface: $d\Omega'_1 = d\Omega(\mathbf{n}'_1)$. Here again, $\mathbf{V}$ is considered as a simple constant "spectator" vector during the average.

(III) We impose that the new frame is actually the rest frame of the particle 1 at the event P_1. Recalling that the Lorentz boost (3.4) brings a particle with velocity $\mathbf{V}$ in the frame $\{x^\mu\}$ at rest in the frame $\{x'^\mu\}$, we see that we must choose

$$\mathbf{V} = \mathbf{v}_1(t) . \tag{3.28}$$

We come back to the original variables in the un-primed frame by using the transformation laws (3.22)-(3.23), in the limit where the field point $\mathbf{x}$ tends to the source point $\mathbf{y}_1(t)$ (because we are located at the event P_1), with $\mathbf{V} = \mathbf{v}_1$ according to (3.28). Note that, in this limit $\mathbf{r}_1 \to \mathbf{0}$, the coordinate time t' of the event Q in the primed frame is equal to the coordinate time τ'_1 of the event P_1. It is important to realize that both the computation of the limit when $\mathbf{r}_1 \to \mathbf{0}$ and the replacement of the vector $\mathbf{V}$ by (3.28) are to be done *after* performing the many partial time differentiations in (3.22)-(3.23). Consider first the primed variable $\mathbf{y}'_{12}$, which is given by (3.22) where we apply the replacement $\mathbf{r}_1 = \mathbf{0}$ (as well as $\mathbf{V} = \mathbf{v}_1$). In (3.22) the $n-1$ partial time derivatives acting on the term proportional to $(\mathbf{V}.\mathbf{r}_1)^n$ will clearly lead to zero in the limit $\mathbf{r}_1 = \mathbf{0}$; indeed, by an argument met previously, there are not "enough" derivatives to make a non-zero contribution. So the variable to be used when coming back to the original frame is

$$\mathbf{y}'_{12} = \left(\mathbf{y}_{12} - \sum_{n=1}^{+\infty} \frac{(-)^n}{c^{2n} n!} \left(\frac{\partial}{\partial t}\right)^{n-1} \left[(\mathbf{V}.\mathbf{r}_2)^n \left(\mathbf{v}_2 - \frac{\gamma}{\gamma+1}\mathbf{V}\right)\right]\right)\Bigg|_{\substack{\mathbf{r}_2 = \mathbf{y}_{12} \\ \mathbf{V} = \mathbf{v}_1}} . \tag{3.29}$$

As indicated by the notation one must implement the replacements of $\mathbf{r}_2$ by $\mathbf{y}_{12}$ (this is equivalent to $\mathbf{r}_1 = \mathbf{0}$) and of $\mathbf{V}$ by $\mathbf{v}_1$ *after* the $n-1$ time differentiations. In the case of the primed velocity of the particle 2, given by (3.23b), we simply have

$$\mathbf{v}'_2 = \left(\frac{1}{\gamma}\mathbf{v}_2 - \mathbf{V} + \frac{1}{\gamma}\sum_{n=1}^{+\infty} \frac{(-)^n}{c^{2n} n!} \left(\frac{\partial}{\partial t}\right)^n \left[(\mathbf{V}.\mathbf{r}_2)^n \left(\mathbf{v}_2 - \frac{\gamma}{\gamma+1}\mathbf{V} \right) \right] \right)\Bigg|_{\substack{\mathbf{r}_2=\mathbf{y}_{12} \\ \mathbf{V}=\mathbf{v}_1}} . \tag{3.30}$$

The formulas (3.29) and (3.30) define, after order-reduction of the accelerations, some functionals $\mathbf{y}'_{12}[\mathbf{y}_{12}; \mathbf{v}_1, \mathbf{v}_2]$ and $\mathbf{v}'_2[\mathbf{y}_{12}; \mathbf{v}_1, \mathbf{v}_2]$ that we use for coming back to the initial frame $\{x^\mu\}$. Clearly, the primed velocity $\mathbf{v}'_1$ of the point 1, at which we perform the regularization, deserves a special treatment. From (3.23a) we obtain

$$\mathbf{v}'_1 = \left(\frac{1}{\gamma}\mathbf{v}_1 - \mathbf{V} + \frac{1}{\gamma}\sum_{n=1}^{+\infty} \frac{(-)^n}{c^{2n} n!} \left(\frac{\partial}{\partial t}\right)^n \left[(\mathbf{V}.\mathbf{r}_1)^n \left(\mathbf{v}_1 - \frac{\gamma}{\gamma+1}\mathbf{V} \right) \right] \right)\Bigg|_{\substack{\mathbf{r}_1=\mathbf{0} \\ \mathbf{V}=\mathbf{v}_1}} . \tag{3.31}$$

Here, there are n time derivatives which is *a priori* enough to make a contribution. The only possibility is to differentiate successively each of the n factors $\mathbf{V}.\mathbf{r}_1$, yielding for each of the terms in the sum $n!$ identical contributions. Hence, we arrive at a much simpler series,

$$\mathbf{v}'_1 = \left(\frac{1}{\gamma}\mathbf{v}_1 - \mathbf{V} + \frac{1}{\gamma}\sum_{n=1}^{+\infty} \left(\frac{\mathbf{V}.\mathbf{v}_1}{c^2}\right)^n \left(\mathbf{v}_1 - \frac{\gamma}{\gamma+1}\mathbf{V} \right) \right)\Bigg|_{\mathbf{V}=\mathbf{v}_1} , \tag{3.32}$$

which can now easily be summed up. The result is

$$\mathbf{v}'_1 = \left(\frac{\frac{1}{\gamma}\mathbf{v}_1 - \mathbf{V} + \frac{\gamma}{\gamma+1}\frac{\mathbf{V}.\mathbf{v}_1}{c^2}\mathbf{V}}{1 - \frac{\mathbf{V}.\mathbf{v}_1}{c^2}} \right)\Bigg|_{\mathbf{V}=\mathbf{v}_1} , \tag{3.33}$$

from which we immediately deduce that the primed velocity of the particle 1 must be zero,

$$\mathbf{v}'_1 = \mathbf{0} . \tag{3.34}$$

This is of course the expected result because the boost velocity was chosen to be equal to the instantaneous velocity of the particle 1 in the un-primed frame at the instant t; however, the details of the above proof constitute a necessary consistency check of the formulas.

(IV) The choice of boost vector $\mathbf{V} = \mathbf{v}_1$, together with the equivalent statement that $\mathbf{v}'_1 = \mathbf{0}$, as well as the expressions (3.29) and (3.30) defining the two functionals $\mathbf{y}'_{12}[\mathbf{y}_{12}; \mathbf{v}_1, \mathbf{v}_2]$ and $\mathbf{v}'_2[\mathbf{y}_{12}; \mathbf{v}_1, \mathbf{v}_2]$, are put into (3.27), which gave the result ${}_1\hat{f}'_0$ of the spherical average in the Hadamard regularization performed in the primed frame. Therefore, the regularized value of F at the point 1 is defined by

$$[F]_1 = \underset{1}{\hat{f}'_0}\Big(\mathbf{y}'_{12}[\mathbf{y}_{12}; \mathbf{v}_1, \mathbf{v}_2]; \mathbf{0}, \mathbf{v}'_2[\mathbf{y}_{12}; \mathbf{v}_1, \mathbf{v}_2]; \mathbf{v}_1 \Big) . \tag{3.35}$$

The new regularization $[F]_1$ acts, like the old one $(F)_1$, as a certain functional of the relative distance $\mathbf{y}_{12}$ and the velocities $\mathbf{v}_1$, $\mathbf{v}_2$. However, in generic cases, $[F]_1$ differs from $(F)_1$ by relativistic terms at least of the order $1/c^2$ [we investigate in Section IV the exact relation between both regularizations to the first relativistic order $1/c^2$]. In the problem of the post-Newtonian equations of motion, we have found [22,24] that the new regularization $[F]_1$ adds some extra terms to the acceleration computed using the regularization $(F)_1$; these new terms are of order 3PN and manage to make the 3PN equations of motion invariant with respect to Lorentz transformations. Indeed, with the regularization $(F)_1$ the Lorentz invariance of the equations of motion would be broken at the 3PN order. Finally, let us introduce as we did in [1] (see also Section II) a delta-pseudo-function associated with the new regularization $[F]_1$. By definition, the "Lorentzian" delta-pseudo-function denoted $\mathrm{Pf}\Delta_1$ [to contrast with the non-invariant one $\mathrm{Pf}\delta_1$ defined by (2.13)] is such that

$$\forall F \in \mathcal{F}, \quad < \mathrm{Pf}\Delta_1, F >= [F]_1 \ , \tag{3.36}$$

where the right side is given by the new regularization (3.35). By definition, we have in the case of the new regularization the same laws for the multiplication as in Section II, for instance

$$FG\,.\,\mathrm{Pf}\Delta_1 = \mathrm{Pf}(F\Delta_1)\,.\,G = \mathrm{Pf}(F\Delta_1)\,.\,\mathrm{Pf}G = \mathrm{Pf}(FG\Delta_1) \ , \tag{3.37}$$

where the pseudo-function $\mathrm{Pf}(F\Delta_1)$ is defined by

$$\forall G \in \mathcal{F} \ , \quad < \mathrm{Pf}(F\Delta_1), G >= [FG]_1 \ . \tag{3.38}$$

This pseudo-function $\mathrm{Pf}(F\Delta_1)$ is at the basis of our proposal for the stress-energy tensor of point-particles in Section V. And, like in the case of $\mathrm{Pf}(F\delta_1)$, we are not allowed to replace this pseudo-function by the product of the regularized value of the function times the delta-pseudo-function, namely

$$\mathrm{Pf}(F\Delta_1) \neq [F]_1 \ \mathrm{Pf}\Delta_1 \ . \tag{3.39}$$

The derivatives of $\mathrm{Pf}\Delta_1$ and $\mathrm{Pf}(F\Delta_1)$ are constructed in the same way as for the original regularization in Section II. Therefore,

$$\forall G \in \mathcal{F} \ , \qquad < \partial_i[\mathrm{Pf}(F\Delta_1)], G >= - < \mathrm{Pf}(F\Delta_1), \partial_i G >= -[F\partial_i G]_1 \ . \tag{3.40}$$

However, the identity (2.5) is not valid in the case of the new regularization, so we do not have a result similar to (2.20) [see (4.13) for the equivalent of (2.5) at the first relativistic order]. For the product of $\mathrm{Pf}(F\delta_1)$ with some $\mathrm{Pf}G$, the Leibniz rule holds:

$$\partial_i[\mathrm{Pf}(F\Delta_1).\,\mathrm{Pf}G] = \partial_i[\mathrm{Pf}(F\Delta_1)].\,\mathrm{Pf}G + \mathrm{Pf}(F\Delta_1).\,\partial_i(\mathrm{Pf}G) \ . \tag{3.41}$$

This is a consequence of the definition (3.40) and the law (3.37).

IV. THE REGULARIZATION AT THE FIRST RELATIVISTIC ORDER

At this point, it is instructive (and useful in practice) to present the complete formulas that define the Lorentzian regularization $[F]_1$ at the level of the first relativistic corrections $1/c^2$, i.e. neglecting all the terms of order $O(1/c^4)$. [Notice that, consistently with Section III, we must consider that the boost vector $\mathbf{V}$ itself is of order $O(1)$, so that for instance the factor $\mathbf{V}^2/c^2$ really represents a small relativistic correction of the order $O(1/c^2)$.] Furthermore, we shall obtain at this $1/c^2$ level a formula linking the new regularization $[F]_1$ to the old one $(F)_1$. Like in Section III, we assume that the function F depends on $\mathbf{x}$ through the two distances $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ only; this implies a relation between the partial derivatives:

$$\partial_i F + \frac{\partial F}{\partial y_1^i} + \frac{\partial F}{\partial y_2^i} = 0 \tag{4.1}$$

(where $\partial_i = \partial/\partial x^i$). We suppose also that F is a Lorentz scalar, *cf* (3.25).

We follow the general specification for the regularization in Section III. We first express the vectorial distances $\mathbf{r}_1$, $\mathbf{r}_2$ and velocities $\mathbf{v}_1$, $\mathbf{v}_2$ in the boosted frame $\{x'^\mu\}$ using the transformation formulas (3.24) restricted to the order $1/c^2$. For the distances, we get

$$\mathbf{r}_1 = \mathbf{r}'_1 - \frac{1}{c^2}(\mathbf{V}.\mathbf{r}'_1)\left[\mathbf{v}'_1 + \tfrac{1}{2}\mathbf{V}\right] + O\left(\frac{1}{c^4}\right) , \tag{4.2a}$$

$$\mathbf{r}_2 = \mathbf{r}'_2 - \frac{1}{c^2}(\mathbf{V}.\mathbf{r}'_2)\left[\mathbf{v}'_2 + \tfrac{1}{2}\mathbf{V}\right] + O\left(\frac{1}{c^4}\right) . \tag{4.2b}$$

The relative distance $\mathbf{y}_{12} = \mathbf{r}_2 - \mathbf{r}_1$ reads as

$$\mathbf{y}_{12} = \mathbf{y}'_{12} + \frac{1}{c^2}\left[-\tfrac{1}{2}(\mathbf{V}.\mathbf{y}'_{12})\mathbf{V} + (\mathbf{V}.\mathbf{r}'_1)\mathbf{v}'_1 - (\mathbf{V}.\mathbf{r}'_2)\mathbf{v}'_2\right] + O\left(\frac{1}{c^4}\right) , \tag{4.3}$$

while, for instance, the relative separation $r_{12} = |\mathbf{y}_{12}|$ is

$$r_{12} = r'_{12}\left(1 + \frac{1}{c^2}\left[-\tfrac{1}{2}(\mathbf{V}.\mathbf{n}'_{12})^2 + \tfrac{r'_1}{r'_{12}}(\mathbf{V}.\mathbf{n}'_1)(\mathbf{v}'_1.\mathbf{n}'_{12}) - \tfrac{r'_2}{r'_{12}}(\mathbf{V}.\mathbf{n}'_2)(\mathbf{v}'_2.\mathbf{n}'_{12})\right]\right) + O\left(\frac{1}{c^4}\right) , \tag{4.4}$$

where $\mathbf{n}'_1 = \mathbf{r}'_1/r'_1$, $\mathbf{n}'_2 = \mathbf{r}'_2/r'_2$ and $\mathbf{n}'_{12} = \mathbf{y}'_{12}/r'_{12}$. For the two velocities, we find

$$\mathbf{v}_1 = \mathbf{v}'_1 + \mathbf{V} + \frac{1}{c^2}\left(\left[-\tfrac{1}{2}\mathbf{V}^2 - \mathbf{V}.\mathbf{v}'_1\right]\mathbf{v}'_1 - \tfrac{1}{2}(\mathbf{V}.\mathbf{v}'_1)\mathbf{V} + (\mathbf{V}.\mathbf{r}'_1)\mathbf{a}'_1\right) + O\left(\frac{1}{c^4}\right) , \tag{4.5a}$$

$$\mathbf{v}_2 = \mathbf{v}'_2 + \mathbf{V} + \frac{1}{c^2}\left(\left[-\tfrac{1}{2}\mathbf{V}^2 - \mathbf{V}.\mathbf{v}'_2\right]\mathbf{v}'_2 - \tfrac{1}{2}(\mathbf{V}.\mathbf{v}'_2)\mathbf{V} + (\mathbf{V}.\mathbf{r}'_2)\mathbf{a}'_2\right) + O\left(\frac{1}{c^4}\right) , \tag{4.5b}$$

where the two accelerations $\mathbf{a}'_1$ and $\mathbf{a}'_2$ are to be replaced, consistently with the approximation, by their Newtonian values : $\mathbf{a}'_1 = -\frac{Gm_2}{r'^2_{12}}\mathbf{n}'_{12} + O\left(\frac{1}{c^2}\right)$ and $\mathbf{a}'_2 = \frac{Gm_1}{r'^2_{12}}\mathbf{n}'_{12} + O\left(\frac{1}{c^2}\right)$. [Notice that in Section III the regularization has been defined regardless of the type of special-relativistic

interaction involved; in the case of electromagnetism, for instance, we should simply replace the accelerations by their Coulombian values in (4.5).]

Next, we substitute the expressions (4.2) and (4.5) into the scalar function $F[\mathbf{r}_1, \mathbf{r}_2; \mathbf{v}_1, \mathbf{v}_2]$ and perform the expansion to the first order. The result is the scalar function $F'[\mathbf{r}'_1, \mathbf{r}'_2; \mathbf{v}'_1, \mathbf{v}'_2; \mathbf{V}]$ in the new frame; thus

$$\begin{aligned}
F'[\mathbf{r}'_1, \mathbf{r}'_2; \mathbf{v}'_1, \mathbf{v}'_2; \mathbf{V}] &= F[\mathbf{r}'_1, \mathbf{r}'_2; \mathbf{v}'_1 + \mathbf{V}, \mathbf{v}'_2 + \mathbf{V}] \\
&+ \frac{1}{c^2}(\mathbf{V}.\mathbf{r}'_1)\left[v'^i_1 + \tfrac{1}{2}V^i\right]\frac{\partial F}{\partial y_1^i} + \frac{1}{c^2}(\mathbf{V}.\mathbf{r}'_2)\left[v'^i_2 + \tfrac{1}{2}V^i\right]\frac{\partial F}{\partial y_2^i} \\
&+ \frac{1}{c^2}\left(\left[-\tfrac{1}{2}\mathbf{V}^2 - \mathbf{V}.\mathbf{v}'_1\right]v'^i_1 - \tfrac{1}{2}(\mathbf{V}.\mathbf{v}'_1)V^i + (\mathbf{V}.\mathbf{r}'_1)a'^i_1\right)\frac{\partial F}{\partial v_1^i} \\
&+ \frac{1}{c^2}\left(\left[-\tfrac{1}{2}\mathbf{V}^2 - \mathbf{V}.\mathbf{v}'_2\right]v'^i_2 - \tfrac{1}{2}(\mathbf{V}.\mathbf{v}'_2)V^i + (\mathbf{V}.\mathbf{r}'_2)a'^i_2\right)\frac{\partial F}{\partial v_2^i} + O\left(\frac{1}{c^4}\right) ,
\end{aligned} \tag{4.6}$$

where we have used $\frac{\partial F}{\partial r_1^i} = -\frac{\partial F}{\partial y_1^i}$ and $\frac{\partial F}{\partial r_2^i} = -\frac{\partial F}{\partial y_2^i}$. Note that, to this order, the partial derivatives in (4.6) can be evaluated at the primed values $\mathbf{r}'_1$, $\mathbf{r}'_2$ and $\mathbf{v}'_1 + \mathbf{V}$, $\mathbf{v}'_2 + \mathbf{V}$, or equivalently at the non-primed ones $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{v}_1$, $\mathbf{v}_2$. Now we pick up in the new frame the term of zeroth order in the expansion when $r'_1 \to 0$, and perform the angular average with respect to the direction $\mathbf{n}'_1$. This yields the functional of the variables $\mathbf{y}'_{12}$, $\mathbf{v}'_1$, $\mathbf{v}'_2$ and $\mathbf{V}$ which has been defined in (3.27). Since these operations of expanding and averaging represent nothing but the Hadamard regularization in the old sense of (2.3), we can denote them by using the parenthesis appropriate for this regularization. Therefore,

$$\begin{aligned}
\hat{\underset{1}{f}}{}'_0\Big(\mathbf{y}'_{12}; \mathbf{v}'_1, \mathbf{v}'_2; \mathbf{V}\Big) = \Bigg(& F[\mathbf{r}_1, \mathbf{r}_1 + \mathbf{y}'_{12}; \mathbf{v}'_1 + \mathbf{V}, \mathbf{v}'_2 + \mathbf{V}] \\
&+ \frac{1}{c^2}(\mathbf{V}.\mathbf{r}_1)\left[v'^i_1 + \tfrac{1}{2}V^i\right]\frac{\partial F}{\partial y_1^i} + \frac{1}{c^2}\Big(\mathbf{V}.\mathbf{r}_1 + \mathbf{V}.\mathbf{y}'_{12}\Big)\left[v'^i_2 + \tfrac{1}{2}V^i\right]\frac{\partial F}{\partial y_2^i} \\
&+ \frac{1}{c^2}\left(\left[-\tfrac{1}{2}\mathbf{V}^2 - \mathbf{V}.\mathbf{v}'_1\right]v'^i_1 - \tfrac{1}{2}(\mathbf{V}.\mathbf{v}'_1)V^i + (\mathbf{V}.\mathbf{r}_1)a'^i_1\right)\frac{\partial F}{\partial v_1^i} \\
&+ \frac{1}{c^2}\left(\left[-\tfrac{1}{2}\mathbf{V}^2 - \mathbf{V}.\mathbf{v}'_2\right]v'^i_2 - \tfrac{1}{2}(\mathbf{V}.\mathbf{v}'_2)V^i + \Big[\mathbf{V}.\mathbf{r}_1 + \mathbf{V}.\mathbf{y}'_{12}\Big]a'^i_2\right)\frac{\partial F}{\partial v_2^i}\Bigg)_1 + O\left(\frac{1}{c^4}\right) .
\end{aligned} \tag{4.7}$$

We have replaced here the vectorial distance $\mathbf{r}'_1$ by the un-primed notation $\mathbf{r}_1$, noticing that $\mathbf{r}'_1$ is the dummy variable with respect to which the regularization proceeds (with this notation $\mathbf{r}'_2$ is replaced by $\mathbf{r}_1 + \mathbf{y}'_{12}$). Following (3.35), the Lorentzian regularization $[F]_1$ is achieved by posing $\mathbf{V} = \mathbf{v}_1$ and $\mathbf{v}'_1 = \mathbf{0}$, as well as $\mathbf{y}'_{12} = \mathbf{y}'_{12}[\mathbf{y}_{12}; \mathbf{v}_1, \mathbf{v}_2]$ and $\mathbf{v}'_2 = \mathbf{v}'_2[\mathbf{y}_{12}; \mathbf{v}_1, \mathbf{v}_2]$, where the latter functionals are defined in the general case by (3.29) and (3.30). It is convenient to obtain first an intermediate formula by setting $\mathbf{V} = \mathbf{v}_1$ and $\mathbf{v}'_1 = \mathbf{0}$ in (4.7), and by replacing into the terms that are already of order $1/c^2$ the primed variables $\mathbf{y}'_{12}$ and $\mathbf{a}'_1$, $\mathbf{a}'_2$ by the un-primed ones. Using also the identity (4.1), we arrive at

$$[F]_1 = \Bigg(F\left[\mathbf{r}_1, \mathbf{r}_1 + \mathbf{y}'_{12}; \mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}'_2\right] + \frac{1}{2c^2}(\mathbf{v}_1.\mathbf{r}_1)v_1^i\partial_i F$$

$$+\frac{1}{c^2}(\mathbf{v}_1.\mathbf{r}_1)\Big[v_1^i\frac{\partial F}{\partial y_1^i}+v_2^i\frac{\partial F}{\partial y_2^i}+a_1^i\frac{\partial F}{\partial v_1^i}+a_2^i\frac{\partial F}{\partial v_2^i}\Big]$$
$$+\frac{1}{c^2}\left(\tfrac{1}{2}(\mathbf{v}_1.\mathbf{v}_2)v_1^i+\Big[\tfrac{1}{2}\mathbf{v}_1^2-\mathbf{v}_1.\mathbf{v}_2\Big]v_2^i+(\mathbf{v}_1.\mathbf{y}_{12})a_2^i\right)\frac{\partial F}{\partial v_2^i}$$
$$+\frac{1}{c^2}(\mathbf{v}_1.\mathbf{y}_{12})\Big[-\tfrac{1}{2}v_1^i+v_2^i\Big]\frac{\partial F}{\partial y_2^i}\Bigg)_1+O\left(\frac{1}{c^4}\right)\ , \tag{4.8}$$

where $\mathbf{y}'_{12}$ and $\mathbf{v}'_2$ in the first term of the right side are given functions of $\mathbf{y}_{12}$, $\mathbf{v}_1$ and $\mathbf{v}_2$ obtained by approximating (3.29) and (3.30) to the first order; we find

$$\mathbf{y}'_{12}=\mathbf{y}_{12}+\frac{1}{c^2}(\mathbf{v}_1.\mathbf{y}_{12})\Big[-\tfrac{1}{2}\mathbf{v}_1+\mathbf{v}_2\Big]+O\left(\frac{1}{c^4}\right)\ , \tag{4.9a}$$
$$\mathbf{v}'_2=-\mathbf{v}_1+\mathbf{v}_2+\frac{1}{c^2}\left(-\tfrac{1}{2}(\mathbf{v}_1.\mathbf{v}_2)\mathbf{v}_1+\Big[-\tfrac{1}{2}\mathbf{v}_1^2+\mathbf{v}_1.\mathbf{v}_2\Big]\mathbf{v}_2-(\mathbf{v}_1.\mathbf{y}_{12})\mathbf{a}_2\right)+O\left(\frac{1}{c^4}\right) \tag{4.9b}$$

(where the acceleration is equal to its Newtonian value). By inserting (4.9) into (4.8) and expanding to order $1/c^2$, it is easily seen that we cancel out exactly the two last terms in the right-hand side of (4.8), so that the result simplifies appreciably:

$$[F]_1=\Bigg(F\left[\mathbf{r}_1,\mathbf{r}_2;\mathbf{v}_1,\mathbf{v}_2\right]+\frac{1}{2c^2}(\mathbf{v}_1.\mathbf{r}_1)v_1^i\partial_iF$$
$$+\frac{1}{c^2}(\mathbf{v}_1.\mathbf{r}_1)\Big[v_1^i\frac{\partial F}{\partial y_1^i}+v_2^i\frac{\partial F}{\partial y_2^i}+a_1^i\frac{\partial F}{\partial v_1^i}+a_2^i\frac{\partial F}{\partial v_2^i}\Big]\Bigg)_1+O\left(\frac{1}{c^4}\right)\ . \tag{4.10}$$

Finally, we recognize in the right side the partial time-derivative,

$$\partial_tF=v_1^i\frac{\partial F}{\partial y_1^i}+v_2^i\frac{\partial F}{\partial y_2^i}+a_1^i\frac{\partial F}{\partial v_1^i}+a_2^i\frac{\partial F}{\partial v_2^i}\ , \tag{4.11}$$

so that our final result writes

$$[F]_1=\left(F+\frac{1}{c^2}(\mathbf{r}_1.\mathbf{v}_1)\Big[\partial_tF+\tfrac{1}{2}v_1^i\partial_iF\Big]\right)_1+O\left(\frac{1}{c^4}\right)\ . \tag{4.12}$$

The result (4.12) displays the first relativistic corrections brought about by our Lorentzian regularization $[F]_1$. As a check of the formula, let us apply it to the case of the special combination $\partial_iF-3\frac{n_1^i}{r_1}F$ which, as we know from (2.5), has no partie finie at the point 1 in the sense of the old regularization. This is no longer true in the sense of the new regularization. Using the equation (4.12) we find instead

$$[\partial_iF]_1=\left[3\frac{n_1^i}{r_1}\Big(1-\frac{1}{c^2}(\mathbf{n}_1.\mathbf{v}_1)^2\Big)F-\frac{1}{c^2}v_1^i\partial_tF\right]_1+O\left(\frac{1}{c^4}\right)\ . \tag{4.13}$$

The check consists of remarking that because of (2.5) we have $(\partial'_iF'-3\frac{n'^i_1}{r'_1}F')_1=0$ in the rest frame of the particle 1, therefore the equation $[\partial'_iF'-3\frac{n'^i_1}{r'_1}F']_1=0$ must hold in any frame by definition of the new regularization. In the frame where the particle velocity is $\mathbf{v}_1$ we have $\mathbf{r}'_1=\mathbf{r}_1+\frac{1}{2c^2}(\mathbf{v}_1.\mathbf{r}_1)\mathbf{v}_1+O(\frac{1}{c^4})$ and $\partial'_i=\partial_i+\frac{1}{c^2}v_1^i\partial_t+\frac{1}{2c^2}v_1^iv_1^j\partial_j+O(\frac{1}{c^4})$. Inserting these relations into the previous equation, and using the fact that F is a scalar, we recover the formula (4.13) after a short computation.

V. THE STRESS-ENERGY TENSOR OF POINT-PARTICLES

With the Lorentzian regularization in hands, we make a proposal for the description of point-like particles in (post-Newtonian approximations of) general relativity. We recall first the general context of the problem. We want to solve the field equations of general relativity by means of analytic post-Newtonian series, with matter source describing appropriately defined point-particles. The stress-energy tensor of the matter source is supposed to be spatially isolated; we recall that, in this case, general relativity admits the Poincaré group as a global symmetry. We assume the existence and unicity of a global harmonic coordinate system, defined by the gauge conditions

$$\partial_\nu h^{\mu\nu} = 0 \,, \tag{5.1a}$$

$$h^{\mu\nu} = \sqrt{g} g^{\mu\nu} - \eta^{\mu\nu} \,, \tag{5.1b}$$

where $g^{\mu\nu}$ denotes the inverse of the covariant metric $g_{\mu\nu}$, and where g is the opposite of its determinant. The harmonic gauge conditions (5.1) introduce a preferred Minkowskian structure, with Minkowski metric given by $\eta^{\mu\nu} = \mathrm{diag}(-1,1,1,1) = \eta_{\mu\nu}$. Thus, the gravitational field can be described in harmonic coordinates by the Lorentzian tensor field $h^{\mu\nu}$ propagating on the Minkowskian background $\eta^{\mu\nu}$. Similarly, one can think of the trajectories of the particles as accelerated world lines in Minkowski space-time. Subject to the conditions (5.1) the Einstein field equations take the form of wave equations on the flat background,

$$\Box h^{\mu\nu} = \tfrac{16\pi G}{c^4} g\, T^{\mu\nu} + \Lambda^{\mu\nu}[h, \partial h, \partial^2 h] \,, \tag{5.2}$$

where the flat d'Alembertian operator is given by $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$. The right-hand side is made of the sum of the matter source term, with spatially compact support, plus the gravitational source term $\Lambda^{\mu\nu}$, given by a certain functional of the field variables $h^{\rho\sigma}$ and its first and second space-time derivatives, and at least of second order in h. A consequence of the harmonicity conditions is that

$$\partial_\nu \left(g\, T^{\mu\nu} + \tfrac{c^4}{16\pi G}\Lambda^{\mu\nu} \right) = 0 \,, \tag{5.3}$$

which is equivalent (through the contracted Bianchi identity) to the covariant conservation of the matter stress-energy tensor $T^{\mu\nu}$,

$$\nabla_\nu T^{\mu\nu} = 0 \,, \tag{5.4}$$

the latter equation being in turn equivalent to

$$\partial_\nu \left(\sqrt{g}\, g_{\lambda\mu}\, T^{\mu\nu} \right) = \tfrac{1}{2}\sqrt{g}\, \partial_\lambda g_{\mu\nu}\, T^{\mu\nu} \,. \tag{5.5}$$

In this section we regard the matter tensor $T^{\mu\nu}$ as a Lorentz tensor defined with respect to the Minkowski metric $\eta_{\mu\nu}$ singled out by our choice of harmonic coordinates.

To define a model for point-like particles, we follow essentially the derivation of the stress-energy tensor of *test* masses moving on a fixed *smooth* background (see e.g. Weinberg

[30] p. 360). However, in the case of "self-gravitating" particles, we do not have a smooth background at our disposal, and the metric becomes singular at the location of the point-masses. Essentially, we shall propose the value of the (post-Newtonian) metric coefficients on each of the particles to be given by the Lorentzian regularization defined in Section III. This entails supposing that the metric coefficients belong to the class of functions $\mathcal{F}$. This is correct up to the 2PN order [19]; however, we know that the expansion of the metric coefficients (in harmonic coordinates) near the particles, instead of being of the type (2.1)-(2.2), involve some logarithms of the distance to the singularities starting at 3PN order. It was shown [22] that, at this order, the logarithms can be considered as some constants and included into the definition of the partie finie; moreover, they can be finally eliminated from the equations of motion by a change of coordinates. This suggests that we might consider more generally the logarithms as some constants, motivating our assumption that $g_{\mu\nu} \in \mathcal{F}$. On the other hand, it is known [22,24] that the constants s_1 and s_2 entering the partie-finie integral (2.6) must be adjusted in order that the equations of motion can be deducible from a Lagrangian, and in particular admit a conserved energy. For these reasons (presence of logarithms, equations of motion not directly admitting an energy), the following derivation of the stress-energy tensor for particles cannot be considered to be a rigorous proof. However, as we shall see, it is nicely consistent with the regularization, and its result satisfying. Our basic assumption is that the dynamics of the particles follows from the variation, with respect to the metric, of the action

$$I_{\text{particle}} = -m_1 c \int_{-\infty}^{+\infty} dt \sqrt{-[g_{\mu\nu}]_1 v_1^\mu v_1^\nu} + 1 \leftrightarrow 2 \; , \tag{5.6}$$

where $v_1^\mu = (c, d\mathbf{y}_1/dt)$ denotes the coordinate velocity of the particle 1 (we consider a two-body system, but the generalization to N bodies is immediate). The crucial point is that the value of $g_{\mu\nu}$ at 1 is assumed to be given by the Lorentzian regularization defined in Section III. We vary the action (5.6) with respect to the metric, i.e. we imagine that $g_{\mu\nu} \in \mathcal{F}$ is subject to an infinitesimal variation $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ and compute the corresponding change in the action. However, we want the variation of the metric to correspond to the same matter system with two singularities 1 and 2. The evident and most natural way to ensure this is to suppose that $\delta g_{\mu\nu} \in \mathcal{F}$. Under the latter variation the regularized value of the metric at the point 1 undergoes the infinitesimal change $[g_{\mu\nu}]_1 \to [g_{\mu\nu}]_1 + [\delta g_{\mu\nu}]_1$. Therefore, the variation of the action (5.6) reads as

$$\delta I_{\text{particle}} = \tfrac{1}{2} m_1 c \int_{-\infty}^{+\infty} dt \, \frac{v_1^\mu v_1^\nu}{\sqrt{-[g_{\rho\sigma}]_1 v_1^\rho v_1^\sigma}} \, [\delta g_{\mu\nu}]_1 + 1 \leftrightarrow 2 \; . \tag{5.7}$$

From the defining property (3.36) of the delta-pseudo-function $\mathrm{Pf}\Delta_1$, we can re-write (5.7) in the equivalent form

$$\delta I_{\text{particle}} = \tfrac{1}{2} m_1 c \int_{-\infty}^{+\infty} dt \, \frac{v_1^\mu v_1^\nu}{\sqrt{-[g_{\rho\sigma}]_1 v_1^\rho v_1^\sigma}} < \mathrm{Pf}\Delta_1, \delta g_{\mu\nu} > + 1 \leftrightarrow 2 \; . \tag{5.8}$$

Now, recall that the duality bracket is defined by the partie finie of the three-dimensional integral [*cf* (2.8)], so the latter expression can be cast into the standard form appropriate to the definition of a stress-energy tensor $T^{\mu\nu}_{\rm particle}$, namely

$$\delta I_{\rm particle} = \tfrac{1}{2}\int_{-\infty}^{+\infty} dt\ < \sqrt{g}\, T^{\mu\nu}_{\rm particle}\, , \delta g_{\mu\nu} > \ . \tag{5.9}$$

The only difference with the standard definition is that the partie finie takes care of the divergencies at the positions of the particles. By comparing (5.8) and (5.9), we readily find that the corresponding stress-energy tensor density is given by

$$\sqrt{g}\, T^{\mu\nu}_{\rm particle} = m_1 c \frac{v_1^\mu v_1^\nu}{\sqrt{-[g_{\rho\sigma}]_1 v_1^\rho v_1^\sigma}}\, {\rm Pf}\Delta_1 + 1 \leftrightarrow 2\ . \tag{5.10}$$

The stress-energy tensor itself comes immediately from the rule of multiplication of pseudo-functions (3.37):

$$T^{\mu\nu}_{\rm particle} = m_1 c \frac{v_1^\mu v_1^\nu}{\sqrt{-[g_{\rho\sigma}]_1 v_1^\rho v_1^\sigma}}\, {\rm Pf}\left(\frac{\Delta_1}{\sqrt{g}}\right) + 1 \leftrightarrow 2\ , \tag{5.11}$$

This tensor takes the same form as the stress-energy tensor of test particles moving on a smooth background, except that the role of the background field is now played by the metric generated by the particles, regularized following the prescription (3.35). Notice in particular that the factor $1/\sqrt{g}$ inside the partie finie sign Pf should not be replaced by its reguralized value at 1 [see (3.39)]. We propose the tensor (5.11) as a model of particles in the post-Newtonian approximation. From the product rules for pseudo-functions, we get the matter source term in the right-hand side of (5.2) as

$$g\, T^{\mu\nu}_{\rm particle} = m_1 c \frac{v_1^\mu v_1^\nu}{\sqrt{-[g_{\rho\sigma}]_1 v_1^\rho v_1^\sigma}}\, {\rm Pf}\,(\sqrt{g}\Delta_1) + 1 \leftrightarrow 2\ . \tag{5.12}$$

The post-Newtonian iteration of the field equations in [22,24] is based on the latter expression of the matter source term.

We now derive the equations of motion of the particle 1 from the covariant conservation of the stress-energy tensor (5.11):

$$\nabla_\nu T^{\mu\nu}_{\rm particle} = 0\ . \tag{5.13}$$

Notice that thanks to the presence of the delta-pseudo-function, we know that the derivative is “ordinary” and satisfies the Leibniz rule in the sense of (3.41). Thus, we can transform $\nabla_\nu T^{\mu\nu}_{\rm particle}$ in the standard way and find that the equation (5.13) is equivalent, like in the case of continuous sources, to the alternative form

$$\partial_\nu\left(\sqrt{g}\, g_{\lambda\mu}\, T^{\mu\nu}_{\rm particle}\right) = \tfrac{1}{2}\sqrt{g}\,\partial_\lambda g_{\mu\nu}\, T^{\mu\nu}_{\rm particle}\ . \tag{5.14}$$

Then, we integrate (5.14) over a closed volume V_1 surrounding the particle 1 exclusively. The role of the three-dimensional integral is played here by the duality bracket defined by (2.8).

Let us denote by $\mathbf{1}_{\mathrm{V}_1}$ the characteristic function of the volume V_1, such that $\mathbf{1}_{\mathrm{V}_1}(\mathbf{x}) = 1$ if $\mathbf{x} \in \mathrm{V}_1$ and $\mathbf{1}_{\mathrm{V}_1}(\mathbf{x}) = 0$ otherwise [notably, $\mathbf{1}_{\mathrm{V}_1}(\mathbf{y}_2) = 0$]. Thus, we consider

$$< \partial_\nu \left(\sqrt{g}\, g_{\lambda\mu}\, T^{\mu\nu}_{\text{particle}}\right), \mathbf{1}_{\mathrm{V}_1} > = < \tfrac{1}{2}\sqrt{g}\, \partial_\lambda g_{\mu\nu}\, T^{\mu\nu}_{\text{particle}}, \mathbf{1}_{\mathrm{V}_1} > \ . \tag{5.15}$$

(Though $\mathbf{1}_{\mathrm{V}_1}$ does not belong to the class $\mathcal{F}$, it is locally integrable on $\mathbb{R}^3$ and we know that the duality bracket applies on such functions as well; see [1].) The partial derivative ∂_ν in the left-hand side is split into a time-derivative and a space-derivative. Following the rule (3.40), the spatial derivative ∂_i is shifted to the right side of the bracket, where it applies on the characteristic function $\mathbf{1}_{\mathrm{V}_1}$. Because of the presence of the delta-pseudo-function, the derivative of $\mathbf{1}_{\mathrm{V}_1}$ is to be taken in an ordinary sense and is zero. Following the rule (9.7) in [1], an analogous reasoning is valid for the time-derivative $\partial_0 = \frac{1}{c}\partial_t$ which can thus simply be put outside the bracket. Thus, we get

$$\frac{d}{cdt}\Big\{< \sqrt{g}\, g_{\lambda\mu}\, T^{\mu 0}_{\text{particle}}, \mathbf{1}_{\mathrm{V}_1} >\Big\} = < \tfrac{1}{2}\sqrt{g}\, \partial_\lambda g_{\mu\nu}\, T^{\mu\nu}_{\text{particle}}, \mathbf{1}_{\mathrm{V}_1} > \ . \tag{5.16}$$

Next, we insert into (5.16) the specific expression (5.10) of the stress-energy density of particles. Because of the presence of the function $\mathbf{1}_{\mathrm{V}_1}$ only the part corresponding to the particle 1 contributes, and we obtain

$$\frac{d}{dt}\Bigg\{\frac{v_1^\mu}{\sqrt{-[g_{\rho\sigma}]_1 v_1^\rho v_1^\sigma}} < \mathrm{Pf}(g_{\lambda\mu}\Delta_1), \mathbf{1}_{\mathrm{V}_1} >\Bigg\} = \tfrac{1}{2}\frac{v_1^\mu v_1^\nu}{\sqrt{-[g_{\rho\sigma}]_1 v_1^\rho v_1^\sigma}} < \mathrm{Pf}(\partial_\lambda g_{\mu\nu}\Delta_1), \mathbf{1}_{\mathrm{V}_1} > \ . \tag{5.17}$$

Finally, the effect of the brackets in both sides of the latter equation is to take the value at the point 1 in the sense of the Lorentzian regularization (3.35). Thereby our final result reads as

$$\frac{d}{dt}\left(\frac{[g_{\lambda\mu}]_1 v_1^\mu}{\sqrt{-[g_{\rho\sigma}]_1 v_1^\rho v_1^\sigma}}\right) = \tfrac{1}{2}\frac{[\partial_\lambda g_{\mu\nu}]_1 v_1^\mu v_1^\nu}{\sqrt{-[g_{\rho\sigma}]_1 v_1^\rho v_1^\sigma}} \ . \tag{5.18}$$

The equations of motion of the particle 1 have the same formal structure as the geodesic equations of a test particle. In separate papers [22,24,25] we use (5.18) to derive explicitely the equations of motion of the two particles at the 3PN approximation.

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APPENDIX A: SOLUTION OF THE EQUATION (3.14)

We are looking for the vector $\mathbf{z}_1$ satisfying the equation

$$\mathbf{z}_1 = \mathbf{y}_1\Big(t - \frac{1}{c^2}\mathbf{V}.(\mathbf{x}-\mathbf{z}_1)\Big) \ , \tag{A1}$$

where $\mathbf{y}_1(t)$ represents a given smooth (C^∞) time-like trajectory and $\mathbf{V}$ a constant vector with norm $|\mathbf{V}| < c$. Clearly, for a given trajectory, the solution $\mathbf{z}_1$ depends on the field point $\mathbf{x}$ as well as on time t. It was shown in the text after (3.15) that the application $\mathbf{x} \to \mathbf{z}_1$ is contracting with fixed point $\mathbf{y}_1$. Here, let us look for the solution $\mathbf{z}_1$ in the form of a function of the coordinates,

$$\mathbf{z}_1 = \mathbf{z}_1(\mathbf{x}, t) \ . \tag{A2}$$

From (A1) we compute the partial derivatives of $\mathbf{z}_1$ with respect to t and x^i, considered to be independent, and readily obtain

$$\frac{\partial \mathbf{z}_1}{\partial x^i} = -\frac{1}{c^2}\left[V_i - \mathbf{V}.\frac{\partial \mathbf{z}_1}{\partial x^i}\right]\mathbf{v}_1\Big(t - \frac{1}{c^2}\mathbf{V}.(\mathbf{x}-\mathbf{z}_1)\Big) \tag{A3a}$$

$$\frac{\partial \mathbf{z}_1}{\partial t} = \left[1 + \frac{1}{c^2}\mathbf{V}.\frac{\partial \mathbf{z}_1}{\partial t}\right]\mathbf{v}_1\Big(t - \frac{1}{c^2}\mathbf{V}.(\mathbf{x}-\mathbf{z}_1)\Big) \ . \tag{A3b}$$

Contracting these equations with the vector $\mathbf{V}$ we can obtain the scalar products $\mathbf{V}.\frac{\partial \mathbf{z}_1}{\partial x^i}$ and $\mathbf{V}.\frac{\partial \mathbf{z}_1}{\partial t}$, and use them back into (A3) with the result that

$$\frac{\partial \mathbf{z}_1}{\partial x^i} = -\frac{1}{c^2}V_i\frac{\mathbf{v}_1}{1-\frac{\mathbf{V}.\mathbf{v}_1}{c^2}} \tag{A4a}$$

$$\frac{\partial \mathbf{z}_1}{\partial t} = \frac{\mathbf{v}_1}{1-\frac{\mathbf{V}.\mathbf{v}_1}{c^2}} \ , \tag{A4b}$$

where the velocity $\mathbf{v}_1$ is evaluated at the instant $t - \frac{1}{c^2}\mathbf{V}.(\mathbf{x}-\mathbf{z}_1)$. In particular, we find that $\mathbf{z}_1$ must be a solution of the following first-order differential equation:

$$\frac{\partial \mathbf{z}_1}{\partial x^i} = -\frac{1}{c^2}V_i\frac{\partial \mathbf{z}_1}{\partial t} \ . \tag{A5}$$

Conversely, let us prove that a vector $\mathbf{z}_1$ that (i) satisfies the differential equation (A5) and (ii) admits $\mathbf{y}_1(t)$ as a *fixed* point, i.e. is such that

$$\mathbf{z}_1\left(\mathbf{y}_1(t), t\right) = \mathbf{y}_1(t) \ , \tag{A6}$$

necessarily satisfies the original equation (A1). Such a $\mathbf{z}_1(\mathbf{x}, t)$ being given, we perform in the equation (A5) the change of variables $(x^i, t) \to (\rho_1^i, \tau_1)$ defined by

$$\rho_1^i = x^i - z_1^i(\mathbf{x}, t) \ , \tag{A7a}$$

$$\tau_1 = t - \frac{1}{c^2}\mathbf{V}.\Big(\mathbf{x} - \mathbf{z}_1(\mathbf{x}, t)\Big) \ . \tag{A7b}$$

Using (A5) it is easy to obtain the laws of transformation of the partial derivatives:

$$\frac{\partial}{\partial\rho_1^i} = \frac{\partial}{\partial x^i} + \frac{1}{c^2} V_i \frac{\partial}{\partial t} \; , \tag{A8a}$$

$$\frac{\partial}{\partial\tau_1} = \frac{\partial}{\partial t} + B^i{}_j \frac{\partial z_1^j}{\partial t} \frac{\partial}{\partial x^i} \; , \tag{A8b}$$

where $B^i{}_j$ denotes the matrix inverse of $A^j{}_k = \delta^j_k + \frac{1}{c^2} V_k \frac{\partial z_1^j}{\partial t}$ (i.e. $A^i{}_j B^j{}_k = \delta^i_k$; in the case considered here where the velocities are strictly less than c the matrix $A^i{}_j$ is a deformation of the unit matrix and thus admits an inverse). Now, under the change of variables (A7) the differential equation (A5) becomes simply

$$\frac{\partial \mathbf{z}_1}{\partial \rho_1^i} = \mathbf{0} \; , \tag{A9}$$

whose general solution is an arbitrary function of the time variable τ_1. Therefore, there must exists a trajectory $\mathbf{Y}_1$ such that

$$\mathbf{z}_1 = \mathbf{Y}_1(\tau_1) = \mathbf{Y}_1\left(t - \frac{1}{c^2}\mathbf{V}.(\mathbf{x} - \mathbf{z}_1)\right) \; . \tag{A10}$$

Imposing now that $\mathbf{y}_1(t)$ is a fixed point for this solution $\mathbf{z}_1$ in the sense of (A6) leads immediately to

$$\mathbf{Y}_1(t) = \mathbf{y}_1(t) \; , \tag{A11}$$

so the equation (A1) is recovered exactly. Thus, solving (A1) is equivalent to solving the differential equation (A5) supplemented by the condition (A6). Notice that from (A1) or equivalently from (A5)-(A6) we find that $\mathbf{z}_1$ tends to the fixed point in the "non-relativistic" limit $c \to +\infty$, i.e.

$$\lim_{c\to+\infty} \{\mathbf{z}_1(\mathbf{x}, t)\} = \mathbf{y}_1(t) \; . \tag{A12}$$

This suggests to look for the solution $\mathbf{z}_1$ in the form of an infinite series of relativistic corrections of successive orders $1/c^{2n}$ [from (A5) we know that $\mathbf{z}_1$ is a function of $1/c^2$]. Thus, taking also into account the limit (A12), we pose

$$\mathbf{z}_1(\mathbf{x}, t) = \mathbf{y}_1(t) + \sum_{n=1}^{+\infty} \frac{1}{c^{2n}} \overset{n}{\mathbf{Z}}_1 (\mathbf{x}, t) \; , \tag{A13}$$

and we look for each one of the unknown coefficients $\overset{n}{\mathbf{Z}}_1 (\mathbf{x}, t)$. By replacing the series (A13) into both sides of the equation (A5) and identifying the factors of the powers of $1/c^2$ on each side we find, for any $n \geq 1$,

$$\frac{\partial \overset{n}{\mathbf{Z}}_1}{\partial x^i} = -V_i \frac{\partial \overset{n-1}{\mathbf{Z}}_1}{\partial t} \; , \tag{A14}$$

with the convention that $\overset{0}{\mathbf{Z}}_1 = \mathbf{y}_1(t)$. The equations (A14) are to be solved using the condition of fixed point $\mathbf{y}_1$ [*cf* (A6)], which implies that, $\forall n \geq 1$,

$$\overset{n}{\mathbf{Z}}_1 \left(\mathbf{y}_1(t), t\right) = \mathbf{0} \; . \tag{A15}$$

The solution of (A14)-(A15) is found by induction over n. As an induction hypothesis suppose that

$$\overset{n-1}{\mathbf{Z}}_1 = \frac{(-)^{n-1}}{(n-1)!} \left(\frac{\partial}{\partial t}\right)^{n-2} \Big[(\mathbf{V}.\mathbf{r}_1)^{n-1} \mathbf{v}_1\Big] \; , \tag{A16}$$

where $\mathbf{r}_1 = \mathbf{x} - \mathbf{y}_1$, and where the partial time derivatives act on t keeping the space coordinate $\mathbf{x}$ fixed: for instance, $\partial \mathbf{r}_1/\partial t = -\mathbf{v}_1$ and $\partial \mathbf{v}_1/\partial t = d\mathbf{v}_1/dt = \mathbf{a}_1$, where $\mathbf{a}_1$ is the acceleration. Notice that (A16) satisfies the condition (A15) because it involves $n-2$ partial time derivatives while there is a factor $(\mathbf{V}.\mathbf{r}_1)^{n-1}$ inside the brackets, so after differentiation there will remain at least one factor $\mathbf{V}.\mathbf{r}_1$ making the result be zero when $\mathbf{x} = \mathbf{y}_1$. Inserting (A16) into the right-hand side of (A14) we obtain the equation to be satisfied for the next-order coefficient,

$$\frac{\partial \overset{n}{\mathbf{Z}}_1}{\partial x^i} = V_i \frac{(-)^n}{(n-1)!} \left(\frac{\partial}{\partial t}\right)^{n-1} \Big[(\mathbf{V}.\mathbf{r}_1)^{n-1} \mathbf{v}_1\Big] \; , \tag{A17}$$

which can be re-written equivalently in the form

$$\frac{\partial \overset{n}{\mathbf{Z}}_1}{\partial x^i} = \frac{\partial}{\partial x^i} \left\{ \frac{(-)^n}{n!} \left(\frac{\partial}{\partial t}\right)^{n-1} \Big[(\mathbf{V}.\mathbf{r}_1)^{n} \mathbf{v}_1\Big] \right\} \; , \tag{A18}$$

showing that the most general solution is necessarily of the type

$$\overset{n}{\mathbf{Z}}_1 = \frac{(-)^n}{n!} \left(\frac{\partial}{\partial t}\right)^{n-1} \Big[(\mathbf{V}.\mathbf{r}_1)^{n} \mathbf{v}_1\Big] + \mathbf{C}(t) \; , \tag{A19}$$

where $\mathbf{C}(t)$ denotes an arbitrary vector depending only on time t. However, this vector must be zero on account of the fact that the result should be zero when $\mathbf{x} = \mathbf{y}_1$. Therefore we have proved by induction that

$$\overset{n}{\mathbf{Z}}_1 = \frac{(-)^n}{n!} \left(\frac{\partial}{\partial t}\right)^{n-1} \Big[(\mathbf{V}.\mathbf{r}_1)^{n} \mathbf{v}_1\Big] \; , \tag{A20}$$

so the vector $\mathbf{z}_1$ solving at once (A5) and (A6), or equivalently (A1), takes the form of the rather interesting infinite series

$$\mathbf{z}_1 = \mathbf{y}_1 + \sum_{n=1}^{+\infty} \frac{(-)^n}{c^{2n} n!} \left(\frac{\partial}{\partial t}\right)^{n-1} \Big[(\mathbf{V}.\mathbf{r}_1)^{n} \mathbf{v}_1\Big] \; , \tag{A21}$$

which constitutes the solution needed for our work in Section III. Furthermore, subtracting $\mathbf{x}$ from this solution and contracting with $\mathbf{V}$ we obtain after a short calculation the quantity τ_1 which was defined in (A7b):

$$\tau_1 = t + \sum_{n=1}^{+\infty} \frac{(-)^n}{c^{2n} n!} \left(\frac{\partial}{\partial t}\right)^{n-1} \Big[(\mathbf{V}.\mathbf{r}_1)^n\Big] \,. \tag{A22}$$

Now, recall that the latter quantity τ_1 is such that $\mathbf{z}_1 = \mathbf{y}_1(\tau_1)$. Therefore, we see that we can find an alternative expression of the vector $\mathbf{z}_1$ by inserting into $\mathbf{y}_1(\tau_1)$ the series expansion (A22) found for τ_1. Using an infinite Taylor expansion we are led to

$$\mathbf{z}_1 = \mathbf{y}_1 + \sum_{p=0}^{+\infty} \frac{1}{(p+1)!} \frac{d^p \mathbf{v}_1}{dt^p} \left(\sum_{n=1}^{+\infty} \frac{(-)^n}{c^{2n} n!} \left(\frac{\partial}{\partial t}\right)^{n-1} \Big[(\mathbf{V}.\mathbf{r}_1)^n\Big] \right)^{p+1} \,. \tag{A23}$$

Each of the terms is composed of $p+1$ sums; accordingly we introduce $p+1$ summation indices n_1, …, n_p, n_{p+1} so that

$$\begin{aligned}\mathbf{z}_1 = \mathbf{y}_1 &+ \sum_{p=0}^{+\infty} \frac{1}{(p+1)!} \frac{d^p \mathbf{v}_1}{dt^p} \sum_{n_1=1}^{+\infty} \cdots \sum_{n_p=1}^{+\infty} \sum_{n_{p+1}=1}^{+\infty} \frac{(-)^{n_1+\ldots+n_{p+1}}}{c^{2(n_1+\ldots+n_{p+1})}} \\ &\times \left(\frac{\partial}{\partial t}\right)^{n_1-1} \left[\frac{(\mathbf{V}.\mathbf{r}_1)^{n_1}}{n_1!}\right] \cdots \left(\frac{\partial}{\partial t}\right)^{n_p-1} \left[\frac{(\mathbf{V}.\mathbf{r}_1)^{n_p}}{n_p!}\right] \left(\frac{\partial}{\partial t}\right)^{n_{p+1}-1} \left[\frac{(\mathbf{V}.\mathbf{r}_1)^{n_{p+1}}}{n_{p+1}!}\right] \,. \end{aligned} \tag{A24}$$

Next we pose $k = n_1 + \ldots + n_p + n_{p+1}$, replace the index n_{p+1} by k, and operate $p+1$ commutations of summations to arrive at

$$\begin{aligned}\mathbf{z}_1 = \mathbf{y}_1 &+ \sum_{k=1}^{+\infty} \frac{(-)^k}{c^{2k}} \sum_{p=0}^{k-1} \frac{1}{(p+1)!} \frac{d^p \mathbf{v}_1}{dt^p} \sum_{n_1=1}^{q_1} \cdots \sum_{n_p=1}^{q_p} \\ &\times \left(\frac{\partial}{\partial t}\right)^{n_1-1} \left[\frac{(\mathbf{V}.\mathbf{r}_1)^{n_1}}{n_1!}\right] \cdots \left(\frac{\partial}{\partial t}\right)^{n_p-1} \left[\frac{(\mathbf{V}.\mathbf{r}_1)^{n_p}}{n_p!}\right] \left(\frac{\partial}{\partial t}\right)^{n_{p+1}-1} \left[\frac{(\mathbf{V}.\mathbf{r}_1)^{n_{p+1}}}{n_{p+1}!}\right] \,, \end{aligned} \tag{A25}$$

in which $n_{p+1} = k - \sum_{i=1}^{p} n_i$ and $q_j = 1 + \sum_{i=j}^{p+1} (n_i - 1)$ (with $1 \le j \le p$). We must identify the latter complicated expression with the simpler form of the vector $\mathbf{z}_1$ given by (A21). From identifying the powers of $1/c^2$ in both expressions we immediately obtain

$$\begin{aligned}\left(\frac{\partial}{\partial t}\right)^{k-1} \left[\frac{(\mathbf{V}.\mathbf{r}_1)^k}{k!} \mathbf{v}_1\right] &= \sum_{p=0}^{k-1} \frac{1}{(p+1)!} \frac{d^p \mathbf{v}_1}{dt^p} \sum_{n_1=1}^{q_1} \cdots \sum_{n_p=1}^{q_p} \\ &\times \left(\frac{\partial}{\partial t}\right)^{n_1-1} \left[\frac{(\mathbf{V}.\mathbf{r}_1)^{n_1}}{n_1!}\right] \cdots \left(\frac{\partial}{\partial t}\right)^{n_p-1} \left[\frac{(\mathbf{V}.\mathbf{r}_1)^{n_p}}{n_p!}\right] \left(\frac{\partial}{\partial t}\right)^{n_{p+1}-1} \left[\frac{(\mathbf{V}.\mathbf{r}_1)^{n_{p+1}}}{n_{p+1}!}\right] \,. \end{aligned} \tag{A26}$$

Finally, from using the binomial formula for the derivative of a product, we can identify in each side of the latter equation the coefficients of each $d^p\mathbf{v}_1/dt^p$, and we arrive at the relation, valid for any p and any $k \ge p+1$,

$$\sum_{n_1=1}^{q_1}\cdots\sum_{n_p=1}^{q_p}\left(\frac{\partial}{\partial t}\right)^{n_1-1}\left[\frac{(\mathbf{V}.\mathbf{r}_1)^{n_1}}{n_1!}\right]\cdots\left(\frac{\partial}{\partial t}\right)^{n_p-1}\left[\frac{(\mathbf{V}.\mathbf{r}_1)^{n_p}}{n_p!}\right]\left(\frac{\partial}{\partial t}\right)^{n_{p+1}-1}\left[\frac{(\mathbf{V}.\mathbf{r}_1)^{n_{p+1}}}{n_{p+1}!}\right]$$
$$= \frac{(p+1)(k-1)!}{(k-1-p)!}\left(\frac{\partial}{\partial t}\right)^{k-p-1}\left[\frac{(\mathbf{V}.\mathbf{r}_1)^{k}}{k!}\right]. \qquad \text{(A27)}$$

The latter relation actually represents a quite general mathematical formula because we have specified nothing about the scalar product $\mathbf{V}.\mathbf{r}_1$. Therefore, the relation (A27) holds in fact in the case of an arbitrary sufficiently differentiable function $f(t)$, so

$$\sum_{n_1=1}^{q_1}\cdots\sum_{n_p=1}^{q_p}\left(\frac{d}{dt}\right)^{n_1-1}\left[\frac{f^{n_1}}{n_1!}\right]\cdots\left(\frac{d}{dt}\right)^{n_p-1}\left[\frac{f^{n_p}}{n_p!}\right]\left(\frac{d}{dt}\right)^{n_{p+1}-1}\left[\frac{f^{n_{p+1}}}{n_{p+1}!}\right]$$
$$= \frac{(p+1)(k-1)!}{(k-1-p)!}\left(\frac{d}{dt}\right)^{k-p-1}\left[\frac{f^{k}}{k!}\right]. \qquad \text{(A28)}$$

The equivalence obtained above between the formula (A1) and the differential equation (A5) together with the auxiliary condition (A6) shows *indirectly* that the mathematical formula (A28) is correct. However, a *direct* proof of this formula has been found by Tanaka, Sasaki and Tagoshi. Here we reproduce their proof in the particular case where $p=1$, so that $q_1=k-1$ and $n_2=k-n$ (where $n\equiv n_1$), in which case the formula reads, for any $k\geq 2$,

$$\sum_{n=1}^{k-1}\left(\frac{d}{dt}\right)^{n-1}\left[\frac{f^{n}}{n!}\right]\left(\frac{d}{dt}\right)^{k-n-1}\left[\frac{f^{k-n}}{(k-n)!}\right] = 2(k-1)\left(\frac{d}{dt}\right)^{k-2}\left[\frac{f^{k}}{k!}\right]. \qquad \text{(A29)}$$

We replace $f(t)$ in (A29) by its Fourier transform, $f(t)=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}e^{i\omega t}\tilde{f}(\omega)$, and readily find that in order to prove the formula (A29) it suffices to prove the statement that the equation

$$\sum_{n=1}^{k-1}\binom{k}{n}\Big(\omega_{(1}+\omega_2+\ldots+\omega_n\Big)^{n-1}\Big(\omega_{n+1}+\ \ldots+\omega_{k)}\Big)^{k-n-1}$$
$$= 2(k-1)\Big(\omega_1+\omega_2+\ldots+\omega_k\Big)^{k-2} \qquad \text{(A30)}$$

holds identically for any family of real frequencies ω_1, ω_2, …, ω_k. Most importantly, the parenthesis around indices in the left side of (A30) indicate the complete symmetrization over the k frequencies ω_1, …, ω_k [in addition, $\binom{k}{n}$ denotes the binomial coefficient]. Let us single out one of the frequencies, for instance ω_k, and re-write (A30) in a form involving an explicit symmetrization over the other $k-1$ frequencies, ω_1, …, ω_{k-1}, only:

$$\sum_{n=1}^{k-1}\binom{k-1}{n}\Big(\omega_{(1}+\ldots+\omega_n\Big)^{n-1}\Big(\omega_{n+1}+\ \ldots+\omega_{k-1)}+\omega_k\Big)^{k-n-1}$$
$$= (k-1)\Big(\omega_1+\omega_2+\ldots+\omega_k\Big)^{k-2} \qquad \text{(A31)}$$

(in which we have simplified a factor 2 in both sides of the equation). Furthermore, let us replace in the latter formula ω_k by some sum $\omega_k+\ldots+\omega_{k+s}$, and symmetrize over the whole set of frequencies ω_1, ..., ω_{k+s}. This yields, for any s,

$$\sum_{n=1}^{k-1}\binom{k-1}{n}\Big(\omega_{(1}+\ldots+\omega_n\Big)^{n-1}\Big(\omega_{n+1}+\ \ldots+\omega_{k+s)}\Big)^{k-n-1}$$
$$=(k-1)\Big(\omega_1+\omega_2+\ldots+\omega_{k+s}\Big)^{k-2}\,. \qquad \text{(A32)}$$

Now we prove that the equation (A30), or equivalently (A31), is true by induction on the integer k. Therefore, our induction hypothesis is that (A31) is correct for *any* $k\leq K$, and from this we want to show that it is correct again for $k=K+1$. Note that from our induction hypothesis we know that (A32) is also correct for any $k\leq K$ and *any* s. Consider the sum defined by the left side of (A31) in the case where $k=K+1$, say

$$S_{K+1}=\sum_{n=1}^{K}\binom{K}{n}\Big(\omega_{(1}+\ldots+\omega_n\Big)^{n-1}\Big(\omega_{n+1}+\ldots+\omega_{K)}+\omega_{K+1}\Big)^{K-n}\,, \qquad \text{(A33)}$$

where we recall that one of the frequencies, i.e. ω_{K+1}, is "artificially" singled out. However, S_{K+1} is also given by half the left-hand side of (A30) and is symmetric in ω_1, ..., ω_{K+1}. We want to show that S_{K+1} is equal to the right-hand side of (A31) with $k=K+1$. To this end, we transform S_{K+1} with the help of the binomial formula, and obtain after a short calculation

$$S_{K+1}=\sum_{l=0}^{K-1}\frac{\omega_{K+1}^l}{l!}\frac{K!}{(K-l)!}\sum_{n=1}^{K-l}\binom{K-l}{n}\Big(\omega_{(1}+\ldots+\omega_n\Big)^{n-1}\Big(\omega_{n+1}+\ldots+\omega_{K)}\Big)^{K-n-l}\,. \qquad \text{(A34)}$$

Now we have two sums over l and n, and it is easy to recognize that the second sum, over n, can be simplified as soon as $l\geq 1$ by means of (A32) which is correct by induction under the condition that $k\leq K$ and for any s. Posing $K-l=k-1$ and $k+s=K$ we see that this condition is realized if and only if $l\geq 1$. After simplification we find

$$S_{K+1}=K\Big(\omega_1+\ldots+\omega_{K+1}\Big)^{K-1}+\Psi_{K+1}\left(\omega_1,\ \ldots,\omega_K\right)\,, \qquad \text{(A35)}$$

where the first term is the result we want to obtain, and where the second term is a certain function of the frequencies ω_1, ..., ω_K but which does *not* depend on ω_{K+1}. The expression of Ψ_{K+1} is given for completeness as

$$\Psi_{K+1}=\sum_{n=1}^{K}\binom{K}{n}\Big(\omega_{(1}+\ldots+\omega_n\Big)^{n-1}\Big(\omega_{n+1}+\ldots+\omega_{K)}\Big)^{K-n}-K\Big(\omega_1+\ldots+\omega_K\Big)^{K-1}\,. \qquad \text{(A36)}$$

Now we use the fact that S_{K+1} is actually fully symmetric with respect to the $K+1$ frequencies $\omega_1, \ldots, \omega_{K+1}$. Therefore the function Ψ_{K+1} must be a pure constant, independent on any ω_n. Furthermore, we know also that S_{K+1} is a homogeneous polynomial of degree $K-1$ in all the $\omega_1, \ldots, \omega_{K+1}$, so this constant must in fact be zero: $\Psi_{K+1} = 0$. Finally we are able to conclude on the desired result,

$$S_{K+1} = K\Big(\omega_1 + \ \ldots + \omega_{K+1}\Big)^{K-1} \ . \tag{A37}$$

Incidentally, notice that the equality $\Psi_{K+1} = 0$ is itself a consequence of the same mathematical formula, since it follows from setting $k = K+1$ and posing $\omega_{K+1} = 0$ in (A31).

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